DEFORMATION QUANTIZATION AND THE FEDOSOV STAR-PRODUCT ON CONSTANT CURVATURE MANIFOLDS OF CODIMENSION ONE

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In this thesis we construct the Fedosov quantization map on the phase-space of a single particle in the case of all finite-dimensional constant curvature manifolds embeddable in a flat space with codimension one. This set of spaces includes the two-sphere and de Sitter (dS)/anti-de Sitter (AdS) space-times. This quantization map was constructed by deformation quantization (DQ) techniques using, in particular, the algorithm provided by Fedosov.

The purpose of this thesis was four-fold. One was to verify that this quantization procedure gave the same results as previous exact analyses of dS/AdS outside of DQ, i.e., standard dS and AdS quantum mechanics. Another was to verify that the formal series used in the conventional treatment converged by obtaining exact and nonperturbative results for these spaces. The third purpose was to illustrate the direct connection between the Fedosov algorithm, star-products, and the Hilbert space formulation of quantization. The fourth was to further develop and understand the technology of the Fedosov algorithm in the case of cotangent bundles, i.e., phase-space.

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PREFACE

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1.0 INTRODUCTION

Quantum theory from its very beginning, Bohr through Schrodinger, Heisenberg, and Dirac up to the very present has been a complicated, much disputed and argued-over theory. Though there is no question about its immense importance in physics because of its immense success in experiments, it remains a somewhat mysterious set of steps. The issues of its physical meaning and interpretation have remained contentious and are still open. Another issue is that the appropriate mathematical formulation of the theory, though basically well established for most physical systems, still has areas not fully developed or understood. It is the purpose of this thesis to explore one of these areas.

The conventional approach to apply quantum theory to a given physical system is to take the relevant classical theory and apply certain rules to make it a 'quantum theory'. Basically one starts with the classical theory in the language of Hamiltonian mechanics where one has a configuration space (usually the physical space) and the momentum space put together—this space is known as the phase-space. (Sometimes this is generalized into what is called a symplectic manifold and even a further generalization called a Poisson manifold.) The quantization procedure then takes all of the variables of the phase-space, i.e., the configuration space variables and the momenta, (i.e., the x's and p's) and turns them into operators that act on an (in general) infinite dimensional linear vector space with quadratic norm, a Hilbert space. Evolution equations, either for the operators (Heisenberg picture) or for the Hilbert space vectors (Schrödinger picture) are then given. This thesis only deals with quantization of finite-dimensional classical theories, but these rules have been generalized to infinite-dimensional classical theories—i.e., to field theories.

While there have been many successful physical applications of quantization, there have been areas of less success: it has unpleasant infinities in calculations, it appears to have many ambiguities when the configuration space is not flat, it often hides coordinate invariance, and it seems to be unsuccessful when applied to general relativity (GR). These, among other reasons, have led many researchers to look at alternative procedures for taking a classical theory and turning it into a 'quantum theory'. One such possible approach is deformation quantization (DQ), which has attracted the attention of many mathematicians who have been interested in the formal meaning of the term 'quantization'.

It is the purpose of this thesis to investigate one outgrowth of DQ called Fedosov quantization. The Fedosov quantization map σ^{-1} is a one-to-one map from *any* C^∞ phase-space function on T^*M (the phase-space or cotangent bundle associated to any manifold M) to an operator that acts linearly on some Hilbert space. This one-to-one correspondence of operators and phase-space functions is an advantage in a quantization procedure because, for example, canonical quantization is not consistent when applied on the set of all C^∞ phase-space functions on T^*M, it is only consistent on some subset of these functions.

Additionally, Fedosov quantization is an interesting and potentially useful method of quantization because it may be able to solve, or at least understand better, some of the difficulties in quantization. For instance, σ^{-1} is a coordinate-free quantization map which means that all quantities involved in the construction are tensorial. Also, it works, at least formally, on the phase-space of finitely many particles on space-time in GR. Fedosov quantization can be done, in general, on an arbitrary symplectic manifold with its symplectic form ω and a definite choice of a connection by means of a perturbative expansion.

We would like to point out two major problems with Fedosov quantization. First, there are ambiguities that make quantization procedures not unique—each one being physically different from the other. Fedosov quantization is just one example that we chose to work with because it is a coordinate-free object. In the end, however, physical arguments or observations should decide or at least narrow the range of the correct quantization maps to be used. A second major research issue is the convergence of the perturbative expansion used in the construction of Fedosov quantization map. The main purpose and results of this present work is to prove, in a variety of cases, that indeed the series that defines σ^{-1} does converge.

This Fedosov quantization map will be applied to the Hamiltonian for geodesic motion of a single free particle $H = p^2 - m^2$ on a class of constant curvature manifolds. We regard this merely as a test to see how the procedure will work, and not the final theory. In the case of Minkowski space, the map σ^{-1} applied to $H = p^2 - m^2$ gives the well-known Klein-Gordon equation and, in the general case, it will suffer many of the same problems. One problem is that some solutions to the Klein-Gordon equation will have negative probability solutions. Another problem is the absence of an observer independent definition of a particle. See appendix A for more details.

1.1 OUTLINE

This thesis is broken into two main parts: chapters 2 and 3 contain the background information needed about Fedosov quantization, and chapters 4 through 6 contain the results (the proofs and computations of statements within chapter 6 are given in appendices B, C, and D).

The Fedosov quantization map σ^{-1} is a direct generalization of the Weyl quantization map. The Weyl quantization map $\mathcal{W}$ is a one-to-one map from *any* C^∞ phase-space function on $T^*\mathbb{R}^n$ (the phase-space or cotangent bundle associated to any manifold topologically $\mathbb{R}^n$) to an operator that acts linearly on some Hilbert space. Fedosov quantization is valid for all configuration spaces and not just ones that are topologically $\mathbb{R}^n$. Therefore, in chapter 2 we define and review the Weyl quantization map $\mathcal{W}$, which is valid only on the phase-space of spaces (or space-times) topologically equivalent to $\mathbb{R}^n$.

The origin of the Fedosov quantization maps comes from a star-product, i.e., associative yet noncommutative products that map two C^∞ phase-space functions to another, denoted by $f * g = h$. This is why we review star-products in chapter 2. The Fedosov quantization map σ^{-1} came from a particular star-product known as the Fedosov star-product which is defined by $f * g := \sigma\left(\sigma^{-1}(f)\,\sigma^{-1}(g)\right)$ where σ is the inverse of σ^{-1}. We show that the map σ^{-1} stands on its own and star-products are never needed to do quantization on any symplectic manifold. In principle, we could construct the star-product and formulate a quantum system using only this star-product on phase-space, but we restrict ourselves to a Hilbert space quantization.

In chapter 3 we explain its definition, properties, and background information as well as show how to use it to quantize a system described by a Hamiltonian. In section 3.1, we give basic definitions, conventions, and notations. Next in section 3.3 we present the algorithm that defines the quantization procedure and the quantization map σ^{-1}.

In section 3.4 we run the quantization procedure on the Hamiltonian for geodesic motion $H = p^2 - m^2$ on a class of constant curvature manifolds of a single free particle and show how, in principle, to get the differential equation associated to a single free particle, the analogue of a Klein-Gordon equation. As was stated before, we are mainly interested in showing that this process, Fedosov quantization, is well-defined and agrees with known results in Frønsdal C. (1965, 1973, 1975a, 1975b), before attempting to formulate the Dirac equation or go on to quantum field theory (which we do not do in this thesis). In section 3.5 we run Fedosov quantization in detail on the simplest of phase-spaces $T^*\mathbb{R}^n$. There are two equations and a quantity called the global bundle connection $\hat{D}$ that are fundamental to the definition of σ^{-1} and in section 3.6, we discuss their physical origin and meaning. Finally in section 3.7, we present a schematic picture of how to extend the procedure to include all smooth functions, not limited to complex or real analytic functions T^*M.

In chapter 4, we present original results by running the algorithm and constructing the Fedosov quantization map σ^{-1} for the two-sphere $\mathbb{S}^2$ in sections 4.1, 4.2, 4.3, and 4.4. In section 4.1, we define our phase-space connection as well as explain now the constraints are dealt with which is **step 1** of the algorithm. **Step 2** of the algorithm is in section 4.2 where we define the Weyl-Heisenberg bundle over the phase-space of the two-sphere. In section 4.3 we construct our globally defined bundle connection $\hat{D}$ as prescribed in **step 3** of the algorithm. The globally defined bundle connection $\hat{D}$ is constructed exactly by the choice of a particular ansatz. For **step 4** in section 4.4, we construct the quantization map σ^{-1} and analyze its image $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C^{\infty}_{\mathcal{A}}\left(T^*\mathbb{S}^2\right)\right)$. Here we see that $\widetilde{\mathcal{A}}_{D,\hat{D}}$ is a constrained version of the Euclidean group with the angular momentum subgroup $\mathbb{SO}\left(3\right)$.

In chapter 5 we present original results by deriving some formulas that aid us in the algorithm. These formulas are specific to phase-spaces of finitely many particles on manifolds. In section 5.1, we derive the cotangent lift of an arbitrary Levi-Civita connection and in section 5.2 we use it as well as the symplectic form to define Weyl Heisenberg bundle. Finally in section 5.3, we derive an ansatz for a solution to the condition in (3.8). We then show that the resulting condition (5.15) is locally integrable by the Cauchy-Kovalevskaya theorem.

In chapter 6, we present original results by running the algorithm and constructing the Fedosov quantization map σ^{-1} for the all constant curvature manifolds embeddable in a flat space of codimension one $M_{C_{p,q}}$. In section 6.1, we review in detail the geometry of these spaces using the analogy with dS/AdS space-times to guide us.

In sections 6.2, 6.3, 6.4, and 6.5 we implement the algorithm. First in section 6.2, we define our phase-space connection and in 6.3 we define the Weyl-Heisenberg bundle over the phase-space. The globally defined bundle connection $\hat{D}$ is constructed in section 6.4 exactly by means of an ansatz. We then construct the Fedosov quantization map σ^{-1} in section 6.5 as prescribed by the algorithm in section 3.3.

The section 6.6 is devoted to constructing the map σ^{-1} for the same manifold with a different embedding. This has no effect on the algebra of observables $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C_A^\infty\left(T^*M_{C_{p,q}}\right)\right)$, which will be computed as $\mathbb{SO}\left(p+1,q+1\right)$, except that the Casimir invariant of the algebra of observables $\widetilde{\mathcal{A}}_{D,\hat{D}}$ (in equation (6.31) in section 6.8) is dependent on the change. In basic terms it will give us flexibility in choosing different representations of $\mathbb{SO}\left(p+1,q+1\right)$.

In section 6.7 we compute the commutators of $\sigma^{-1}\left(x^\mu\right)$ and $\sigma^{-1}\left(p_\mu\right)$. Next in section 6.8, we reorganize the generators of $\widetilde{\mathcal{A}}_{D,\hat{D}}$, i.e., $\sigma^{-1}\left(x^\mu\right)$ and $\sigma^{-1}\left(p_\mu\right)$, and find that $\widetilde{\mathcal{A}}_{D,\hat{D}} = \mathbb{SO}\left(p+1,q+1\right)$. Using the program in 3.4, in sections 6.9 (the dS/AdS case) and 6.10 (a more general case) we find the differential equation on $\phi\left(x\right) = \langle x|\phi\rangle$ coming from $\sigma^{-1}\left(H\right)|\phi\rangle = 0$ where $H = p^2 - m^2$ is the Hamiltonian associated to geodesic motion of a single free particle on $M_{C_{p,q}}$.

The conclusions are presented in chapter 7.

2.0 WEYL QUANTIZATION AND STAR-PRODUCTS

In this chapter we review Weyl quantization and star-products, as well as how they relate to Fedosov quantization. Weyl quantization is a coordinate-free quantization that is valid only on phase-spaces with configuration spaces that are topologically $\mathbb{R}^n$. Fedosov quantization is a direct and natural generalization of Weyl quantization on an arbitrary phase-space T^*M. However, in the case when the configuration space is topologically $\mathbb{R}^n$ Fedosov quantization is Weyl quantization. Star-products are reviewed here because a particular star-product, known as the Fedosov star-product, defined the original Fedosov quantization map by the isomorphism σ^{-1}, i.e., $f *_F g = \sigma\left(\sigma^{-1}(f)\,\sigma^{-1}(g)\right)$.

A quantization map Q is a map that tries to assign to each phase-space function f an operator $Q(f)$ (also denoted by $\hat{f}$) which acts on an appropriate Hilbert space. Canonical quantization attempts to do this but fails as was shown by the theorem of Groenewold and van Howe (see Tillman P. 2006a)—it only is consistent on a subset of C^∞ functions on phase-space $C^\infty(T^*M)$. In contrast, the Weyl quantization map, which we call $\mathcal{W}$, successfully maps any C^∞ phase-space function f to a unique operator $\mathcal{W}(f)$, but $\mathcal{W}$ works only on the phase-space of any space which is topologically $\mathbb{R}^n$.

The generalization to the n-dimensional case is straightforward once you know how the 1-dimensional case works so we consider only the 1-dimensional case. Let $\hat{x}$ and $\hat{p}$ be Hilbert space operators that satisfy the commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar \quad , \quad [\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$$

The map is defined to be:

$$\mathcal{W}(f(x,p)) := \int d\xi d\eta\; e^{-i(\xi\hat{x}+\eta\hat{p})}\tilde{f}(\xi,\eta) = \hat{f}(\hat{x},\hat{p}) \tag{2.1}$$

which was first formulated by Weyl (see Weyl H. 1931).

Shortly afterwards, Wigner E. (1932) wrote the inverse map $\mathcal{W}^{-1}$:

$$\mathcal{W}^{-1}\left(\hat{f}\left(\hat{x},\hat{p}\right)\right) := \hbar \int dy \left\langle x+\hbar y/2\right| \hat{f} \left|x-\hbar y/2\right\rangle e^{-ipy} = f\left(x,p\right) \tag{2.2}$$

See appendix J for properties of these maps.

Groenewold H. (1946) (and later in Moyal J. 1949) investigated the formula $\mathcal{W}^{-1}\left(\mathcal{W}\left(f\right)\mathcal{W}\left(g\right)\right)$ and found the result:

$$\mathcal{W}^{-1}\left(\mathcal{W}\left(f\right)\mathcal{W}\left(g\right)\right) = f \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^{\mu}}\frac{\overrightarrow{\partial}}{\partial p_{\mu}} - \frac{\overleftarrow{\partial}}{\partial p_{\mu}}\frac{\overrightarrow{\partial}}{\partial x^{\mu}}\right)\right] g$$

The operator between f and g defines an associative, noncommutative product of any two C^{∞} phase-space functions f and g defined by:

$$\begin{aligned} f * g &= \mathcal{W}^{-1}\left(\mathcal{W}\left(f\right)\mathcal{W}\left(g\right)\right) \\ &= f \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^{\mu}}\frac{\overrightarrow{\partial}}{\partial p_{\mu}} - \frac{\overleftarrow{\partial}}{\partial p_{\mu}}\frac{\overrightarrow{\partial}}{\partial x^{\mu}}\right)\right] g \\ &= fg + \frac{i\hbar}{2} f\left(\frac{\overleftarrow{\partial}}{\partial x^{\mu}}\frac{\overrightarrow{\partial}}{\partial p_{\mu}} - \frac{\overleftarrow{\partial}}{\partial p_{\mu}}\frac{\overrightarrow{\partial}}{\partial x^{\mu}}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^{2} f\left(\frac{\overleftarrow{\partial}}{\partial x^{\mu}}\frac{\overrightarrow{\partial}}{\partial p_{\mu}} - \frac{\overleftarrow{\partial}}{\partial p_{\mu}}\frac{\overrightarrow{\partial}}{\partial x^{\mu}}\right)^{2} g + \cdots \end{aligned} \tag{2.3}$$

This product is called the Groenewold-Moyal star-product (for properties of this star-product see appendix H). It is a coordinate-free object because the Poisson bracket is a tensor, however this star-product is not unique. Even though it is a coordinate-free object, the form of it is not left invariant in all coordinates. In some other coordinates $X^{\mu}\left(x,p\right)$ and $P_{\mu}\left(x,p\right)$ we could define another, different Groenewold-Moyal star-product:

$$f \tilde{*} g = f \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial X^{\mu}}\frac{\overrightarrow{\partial}}{\partial P_{\mu}} - \frac{\overleftarrow{\partial}}{\partial P_{\mu}}\frac{\overrightarrow{\partial}}{\partial X^{\mu}}\right)\right] g$$

Even when the two coordinates (x,p) and (X,P) are related by a canonical transformation (i.e., one that preserves the Poisson bracket) the two stars, in general, $*$ and $\tilde{*}$ will be different.

For convenience, we write the products:

$$x^\mu * x^\nu = x^\mu x^\nu \quad , \quad p_\mu * p_\nu = p_\mu p_\nu \quad , \quad x^\mu * p_\nu = x^\mu p_\nu + \frac{i\hbar}{2}\delta^\mu_\nu \quad , \quad p_\nu * x^\mu = x^\mu p_\nu - \frac{i\hbar}{2}\delta^\mu_\nu$$

Using the above relations, it is easy to compute the familiar commutators:

$$[x^\mu, p_\nu]_* = i\hbar\delta^\mu_\nu \quad , \quad [x^\mu, x^\nu]_* = 0 = [p_\mu, p_\nu]_*$$

where the bracket is defined as $[f, g]_* := f * g - g * f$.

Using the isomorphism $\mathcal{W}$, quantum mechanics in $\mathbb{R}^n$ can be reformulate entirely in phase-space using star-products without ever mentioning Hilbert spaces (see Hirshfeld A. and Henselder P. 2002a). There are certain conceptual advantages including that observables do not change in quantization, i.e., they are the same phase-space functions they were in the classical theory. Additionally, concepts of coordinate transformations, topology, etc. (see Tillman P. 2006b) can be extended conceptually more naturally than using operators. It is much more complicated to obtain these concepts using operator methods, for example see Connes A. (1992).

Although this thesis is mainly concerned with constructing a quantization map σ^{-1} on a class of constant curvature manifolds $M_{C_{p,q}}$, i.e., the generalization of the map $\mathcal{W}$, there is a fundamental connection between operator algebras and star-products (see Dito G. and Sternheimer D. 2002). There are various insights to be had by this connection, however we are concerned with only one: the Fedosov star-product and its connection to Hilbert space quantum mechanics.

A general star-product is a pseudodifferential operator, i.e., an operator of the form:

$$f * g = \sum_j^\infty (i\hbar)^j C_j (f, g)$$

for any two phase-space functions f and g, where $C_j (\cdot, \cdot)$ is some bi-differential operator, i.e., $C_j (f, g)$ contains derivatives of both f and g in general for each j.

A star-product is a very complicated object because of the infinite order of derivatives. In fact, it is this "infinite orderness" which causes many difficulties in working with star-products and why we need such complex tools in their construction and classification. A related problem is that the map σ^{-1} is written only as a series in $\hbar$ so convergence of this map must be contended with. Another major problem is that given any phase-space there is no unique star-product, i.e., there are many of them.

The rest of this thesis will be devoted to constructing and describing the Fedosov quantization map σ^{-1} as it is the hardest and most important part of the construction of the Fedosov star-product. However, the map σ^{-1} stands on its own as a procedure for quantization on an arbitrary symplectic manifold. It gives a one-to-one map from functions on the symplectic manifold to linear Hilbert space operators.

In this thesis we will restrict the focus onto phase-spaces of finite dimensional manifolds and not symplectic manifolds because these spaces are the most relevant for the type of physics we are interested in. Fedosov has the construction in the form of an algorithm which we review in section 3.3

3.0 FEDOSOV QUANTIZATION

This chapter contains a reviews the basic ideas, tools, and properties of Fedosov quantization as well as the algorithm itself. First in section 3.1, we review notations, basic definitions, and conventions and in section 3.2 we describe the properties of Fedosov quantization and background information. In section 3.3, we outline the algorithm including noting places where we diverge from Fedosov's original algorithm making it clear what he did and what we did in this thesis.

Following the algorithm, in section 3.4, we describe the quantization of the geodesic Hamiltonian H assuming the Fedosov quantization map σ^{-1} has been constructed. If we force this Hamiltonian to determine a condition on the set of allowed states, i.e., $\sigma^{-1}(H)|\phi\rangle = 0$, then we describe how we would in principle—and later in this thesis we will provide examples of—obtain a differential equation on $\phi(x) := \langle x|\phi\rangle$, similar to a Schrödinger equation or a Klein-Gordon equation.

We give a simple example in section 3.5 by running the program for the phase-space of a single particle on $\mathbb{R}^n$. This is to better understand this whole process of first constructing the Fedosov quantization map σ^{-1} then using it to obtain equations that describe real physical situations, like $\sigma^{-1}(H)|\phi\rangle = 0$.

In the definition of σ^{-1} there are two fundamental conditions appear, equations (3.8) and (3.12), as well as a quantity called the global bundle connection $\hat{D}$. To justify their presence, in section 3.6 we explain their physical origin and meaning of two fundamental conditions appearing in the construction of σ^{-1}, equations (3.8) and (3.12), as well as a quantity called the global bundle connection $\hat{D}$. We end this chapter in section 3.7 on a technical note describing how the Fedosov star-product is well-defined on all smooth functions and not just real/complex analytic ones.

3.1 NOTATIONS AND BASIC DEFINITIONS

Before we go into the algorithm, we need notations and definitions of some basic terms:

Def. The **cotangent space** T_x^*M of a manifold M at the point $x \in M$ is the vector space of all possible momenta p_μ.

Def. The **cotangent bundle** or **phase-space** of M is $T^*M = \cup_{x\in M} T_x^*M$ of a manifold M, i.e., it is the union of all cotangent spaces at all points $x \in M$. A point in this space is represented by (x, p).

Def. The **tangent space** T_xM of a manifold M at the point $x \in M$ is the vector space of all possible linear maps from $T^*M \to \mathbb{R}$ an arbitrary element is denoted by the canonical pairing $v\,(p) = v^\mu p_\mu \in \mathbb{R}$.

Def. The **tangent bundle** of M is $TM = \bigcup_{x\in M} T_xM$ of a manifold M, i.e., it is the union of all tangent spaces at all points $x \in M$. A point in this space is represented by (x, v).

Def. The **configuration space** of any cotangent bundle T^*M is the manifold M.

Def. Let M' be any space then we define $C^\infty\,(M')$ to be the set of smooth real-valued functions from $M' \to \mathbb{R}$.

Def. Let M' be any space then we define $C^\infty_{\mathbb{C}}\,(M')$ to be the set of smooth complex-valued functions from $M' \to \mathbb{C}$.

Def. Let M' be any space then we define $C^\infty_A\,(M')$ to be the set of analytic complex-valued functions from $M' \to \mathbb{C}$.

Def. Let M be a manifold. We say something is **coordinate-free** when it is made out of tensors.

Like Maxwell's equations, the form a coordinate-free object could change from one coordinate system to another. "Coordinate-free" means that in any well-defined coordinate system, the coordinate-free object is well-defined as well.

Def. The wedge product $\wedge$ is defined to be the completely antisymmetric tensor product:

$$\alpha_1 \wedge \cdots \wedge \alpha_n := \frac{1}{n!}\left(\begin{array}{c} \alpha_1 \otimes \cdots \otimes \alpha_n + (\text{even permutations in } \alpha_j) \\ -\,(\text{odd permutations in } \alpha_j) \end{array}\right)$$

Def. The vee product $\vee$ is defined to be the completely symmetric tensor product:

$$\alpha_1 \vee \cdots \vee \alpha_n := \frac{1}{n!}\left(\alpha_1 \otimes \cdots \otimes \alpha_n + (\text{all permutations in } \alpha_j)\right)$$

Def. A **symplectic manifold** (N, ω) is a manifold equipped with a closed two-form ω, which is non-degenerate (i.e., at all points ω_{AB} has an inverse ω^{AB} st. $\omega^{AB}\omega_{BC} = \delta^A_C$). This is called the symplectic form $\omega = \omega_{AB} dq^A \wedge dq^B$. The symplectic form is the inverse of the Poisson bracket $\omega^{AB} \frac{\partial}{\partial q^A} \wedge \frac{\partial}{\partial q^B}$.

Def. The **Hamiltonian vector-field** associated to a Hamiltonian H is defined as:

$$\hat{H} := \omega^{AB} \frac{\partial H}{\partial q^A} \frac{\partial}{\partial q^B}$$

Def. Let Θ^A be some basis of T^*N. For example, in this thesis we will use the particular basis $\Theta^A = dq^A$.

Def. The **space of one-forms** on a symplectic manifold (N, ω), Λ is defined by the set of all elements of the form $f_B dq^B$ where $f_B \in C^\infty(N)$.

Def. The **space of n-forms** on a symplectic manifold (N, ω), $\Lambda^n = \Lambda \wedge \cdots \wedge \Lambda$ is defined by the set of all elements of the form $\theta_1 \wedge \cdots \wedge \theta_n$ where $\theta_j \in \Lambda$ for each j.

Def. A tensor of rank $\binom{n}{k}$ on M is any element of the form:

$$T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_n}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_n} \in \left(\bigotimes_{i=1}^{n} TM\right) \otimes \left(\bigotimes_{i=1}^{k} T^*M\right)$$

Def. A tensor of rank $\binom{n}{k}$ on N is any element of the form:

$$T^{A_1 \cdots A_n}{}_{B_1 \cdots B_n} \frac{\partial}{\partial q^{A_1}} \otimes \cdots \otimes \frac{\partial}{\partial q^{A_n}} \otimes dx^{B_1} \otimes \cdots \otimes dx^{B_n} \in \left(\bigotimes_{i=1}^{n} TN\right) \otimes \left(\bigotimes_{i=1}^{k} T^*N\right)$$

It is well-known that all phase-spaces are symplectic manifolds. Consider $T^*\mathbb{R}^n$, the phase-space of $\mathbb{R}^n$. Choose the coordinates of the configuration space $\mathbb{R}^n$ to be x^μ then there exists canonical momentum associated to these coordinates p_μ. In these coordinates (called Darboux coordinates) of phase-space (x, p) the symplectic form is $\omega = dp_\mu dx^\mu = dp_1 dx^1 + \cdots + dp_n dx^n$. The Poisson bracket is then $\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\mu}$.

For this thesis, let M be a space-time. It is a fact for any M that T^*M is always equipped with a nondegenerate closed two-form ω which is the inverse of the Poisson bracket tensor $\omega^{AB} \frac{\partial}{\partial q^A} \wedge \frac{\partial}{\partial q^B}$.[1] This is a straightforward generalization of the above example in $\mathbb{R}^n$ because

[1] The Poisson bracket tensor has two upstairs indices so is a $(2, 0)$ tensor and the symplectic form is a $(0, 2)$ tensor.

we always have canonical momenta associated to each choice of coordinates x^μ. Therefore, every phase-space is a symplectic manifold. The symplectic form in some local coordinates (x, p) is $\omega = dp_\mu dx^\mu$ where x is the coordinate on M and p is the canonical momentum conjugate to x. Also, on every phase-space we can define a phase-space connection D (we define ours later) which we will need for Fedosov quantization. We define the Fedosov triple by (T^*M, ω, D).

For the construction of the Fedosov quantization map σ^{-1} all we need are a symplectic manifold (N, ω) equipped with a connection D, i.e., a Fedosov triple (N, ω, D). The construction, as we will see in Section 3.3, is in the form of an algorithm which constructs σ^{-1} order-by-order in increasing powers of $\hbar$. The convergence properties of this series in general is unknown and may be in general problematic.

Notation 1: We let the upper-case Latin indices A, B, C etc. be numerical phase-space indices running from $1, \ldots, 2n$. Raising and lowering upper-case Latin indices will be done by the symplectic form ω_{AB}:

$$v^A := \omega^{AB} v_B \quad , \quad w_A := \omega_{AB} w^B \quad , \quad \omega^{AB}\omega_{BC} = \delta^A_C$$

We must be careful about the order of the indices of ω^{AB} and ω_{AB} when we raise and lower because $\omega^{AB} = -\omega^{BA}$ and $\omega_{AB} = -\omega_{BA}$, so that $\omega^{BA} v_B = -v^A$ given the above definition.

Notation 2: We let lower-case Greek indices be numerical configuration space indices running from $1, \ldots, n$. Lower-case Latin indices will be reserved for abstract configuration space indices. Raising and lowering of these indices will be done with some given metric $g_{ab}(x)$.

$$v_a := g_{ab} v^b \quad , \quad w^a := g^{ab} w_b \quad , \quad g_{ab} g^{bc} = \delta^c_a$$

Notation 3: Since writing $\wedge$ and $\vee$ will become cumbersome we will make the convention that we will not write them. We trust that it will be obvious when we mean one or the other. For example, the metric always uses the symmetric tensor product $g = g_{\mu\nu} dx^\mu \vee dx^\nu$ and the symplectic form always uses the antisymmetric tensor product $\omega = \omega_{AB}\Theta^A \wedge \Theta^B$. However, we simply write them $g = g_{\mu\nu} dx^\mu dx^\nu$ and $\omega = \omega_{AB}\Theta^A\Theta^B$.

Notation 4: A series of the form:

$$f(v) = f_{(1)j_1} v^{j_1} + f_{(2)j_1 j_2} v^{j_1} v^{j_2} + \cdots + f_{(l)j_1 \cdots j_l} v^{j_1} \cdots v^{j_l} + \cdots$$

where the Einstein summation convention is implied on each term will be represented by:

$$f(v) = \sum_{(l)} f_{(l)j_1 \cdots j_l} v^{j_1} \cdots v^{j_l}$$

where the sum over (l) means that we add all of these monomials all with different coefficients $f_{(l)j_1 \cdots j_l}$ for each l.

3.2 PROPERTIES OF FEDOSOV QUANTIZATION AND BACKGROUND

The Fedosov quantization map, called σ^{-1}, is constructed by an algorithm given in Fedosov B. (1996). The map σ^{-1} takes a function f on a general phase-space and associates a unique linear Hilbert space operator $\hat{f} = \sigma^{-1}(f)$ to it.

The Fedosov star-product of any two phase-space functions is then defined by:

$$f *_F g := \sigma\left(\sigma^{-1}(f)\, \sigma^{-1}(g)\right) \tag{3.1}$$

analogously to the definition of the Groenewold-Moyal star-product (2.3).

With convergence issues aside, the properties of the Fedosov star are (see Fedosov B. 1996, and Tillman P. and Sparling G. 2006a, 2006b):

1. It is a coordinate-free object.
2. It can be constructed on all symplectic manifolds (including all phase-spaces) perturbatively in powers of $\hbar$.
3. It assumes no symmetries.
4. It gives a precise and well-defined classical limit $\hbar \to 0$.
5. It is equivalent to an operator formalism by a Weyl-like quantization map.
6. When the configuration space is topologically $\mathbb{R}^n$ then the Fedosov star is the Groenewold-Moyal star.

In this thesis we will restrict our focus to phase-spaces of finite dimensional manifolds, denoted by T^*M. The reason for this choice is that we are interested in quantization of a single particle, which is always in a finite dimensional phase-space.

We chose Fedosov quantization because it provides a very natural geometric way of extending the quantization rules for flat phase-spaces to rules for curved ones. The two important conditions defining the construction of the Fedosov quantization map σ^{-1} have solid physical arguments for their presence. In addition, one could define the Fedosov star-product once given the map σ^{-1} and its known projection σ, however everything could be done without ever defining any star-products.

On major problem with Fedosov quantization and deformation quantization as a whole is that most formulas are written as a formal series in $\hbar$, therefore, in general, convergence of these formal series may be problematic. Another problem with these maps is that they are not unique, i.e., there are many of them.

One important fact is that the Fedosov quantization map σ^{-1} only needs a phase-space with its symplectic form and some choice of phase-space connection for its construction. This fact is significant because we could, in principle, construct a map σ^{-1} that is independent of the choice of a metric. Such a choice may be of particular use in the context of GR because in GR the metric becomes a dynamical quantity. However, we could construct a map $\tilde{\sigma}^{-1}$ that does depend on the metric, which we do in this thesis. This quantization map $\tilde{\sigma}^{-1}$, dependant on the metric, is c-equivalent to another σ^{-1} which is not. This c-equivalence is defined as:

Def. Two formal star-products $*_1$ and $*_2$ are **c-equivalent** (c stands for cohomology) if there exists some isomorphism T of the form:

$$T = Id + \hbar T_1 + \cdots = \sum_{j=0}^{\infty} \hbar^j T_j$$

where T_j is a differential operator (possibly containing derivatives of all order) for each j such that:

$$f *_2 g = T^{-1}\left(T\left(f\right) *_1 T\left(g\right)\right) \tag{3.2}$$

Def. Given any two quantization maps σ^{-1} and $\tilde{\sigma}^{-1}$. These maps are defined to be c-equivalent if their associated star-products $*_1$ and $*_2$ defined by:

$$f *_1 g := \sigma\left(\sigma^{-1}(f)\,\sigma^{-1}(g)\right)$$

$$f *_2 g := \tilde{\sigma}\left(\tilde{\sigma}^{-1}(f)\,\tilde{\sigma}^{-1}(g)\right)$$

are c-equivalent.

However, we are not sure of the physical consequences of these mathematical equivalences except that in general the algebra of observables are isomorphic, however each spectra of observables may be different (see Hirshfeld A. and Henselder P. 2002a). These equivalences tell us that σ^{-1} on the phase-space of M with one metric g is a well defined quantization map on the phase-space of M with another metric $\tilde{g}$. Even if $T^*M' \subset T^*M$, but $\dim T^*M = \dim T^*M'$ (codimension is zero), the quantization map σ^{-1} is well-defined on T^*M'. For example, given a quantization map σ^{-1} on $\mathbb{R}^n$, the same map could be used in anything embeddable in $\mathbb{R}^n$ as long as the dimensions are the same. Important examples include both the Schwarzschild and Kerr-Newman space-times (topologically Schwarzschild is $\mathbb{R}^4 - \mathbb{R}$ and Kerr-Newman is $\mathbb{R}^4 - S^1 \times \mathbb{R}$).

A physical principle could select a correct map σ^{-1}, however in finite particle quantum mechanics on phase-spaces, it is our opinion that the ambiguities are there to stay. In QFT on symplectic or Poisson manifolds the story maybe entirely different. In fact, it is argued in Hirshfeld A. and Henselder P. (2002b) that in QFT a unique choice of a star-product might be made on the grounds that the star-product be divergence free.

3.3 THE ALGORITHM

In this section we provide a brief outline of the algorithm that is used to construct the Fedosov quantization map and star-product. We make it clear in places where we diverge from Fedosov's original algorithm.

Because of complications in chapters 4 and 6, in this section we want to illustrate what the algorithm does without unnecessary complications. In particular, the results in chapters 4 and 6 are complicated by the constraints due to the embedding of these manifolds. Specifically we are using a $(n+1)$-dimensional coordinate patch to describe a n-dimensional manifold. Constraints on the $2\,(n+1)$ coordinates of phase-space are needed and must be carried through the algorithm. In this section no constraints are used.

Before we begin the algorithm, we summarize the basic idea for each of the five steps in the algorithm in very basic terms. The starting ingredients of the algorithm are the symplectic form and some phase-space connection. **Step 1** in the algorithm is choosing a phase-space connection. For this thesis we will use as our phase-space connection the 'cotangent lift' of the Levi-Civita connection on the configuration spaces/space-times. In **step 2** we put a matrix algebra of linear Hilbert space operators at each point in the phase-space. This creates an algebra, called the Weyl-Heisenberg bundle E, which will contain our observables. First, in **step 3** we need to define a new connection $\hat{D}$ by (3.8) that is fundamental in the selection of our observables in step 4. In **step 4** we associate a unique operator in E to every phase-space function using, in part, the fundamental condition (3.12). This association defines our desired quantization map σ^{-1}. At this point we are done with Fedosov quantization, however in an optional **step 5** you could construct the Fedosov star-product using the maps σ^{-1} and σ, which were defined in the previous steps.

3.3.1 Step 1: Defining the Phase-Space Connection

Let (N, ω) be a symplectic manifold and let q^A be a set of coordinates in the neighborhood of $q \in N$. We define a phase-space connection D on N by:

$$Df = df = \frac{\partial f}{\partial q^A} dq^A$$

$$D \otimes \Theta^A = \Gamma^A{}_B \otimes \Theta^B = \Gamma^A{}_{BC} \Theta^C \otimes \Theta^B \tag{3.3}$$

where Θ^A (for example let $\Theta^A = dq^A$) is a basis of one-forms in the cotangent bundle of N, denoted by T^*N, and $\omega = \omega_{AB}\Theta^A \wedge \Theta^B$ (i.e., the inverse of the Poisson bracket tensor $\omega^{AB}\frac{\partial}{\partial q^A} \wedge \frac{\partial}{\partial q^B}$). The symbol $\Gamma^A{}_{BC}$ is defined by the conditions that D preserves the symplectic form ω ($D \otimes \omega = 0$) and that D be torsion-free ($D \wedge Df = 0$):

$$D \otimes \omega = 0 \quad , \quad D \wedge Df = 0 \tag{3.4}$$

We can extend to higher order tensors by using the Leibnitz rule and the fact that D commutes with contractions.

Darboux's Theorem: In the neighborhood of every point $q \in N$ there always exist a set of coordinates, called Darboux coordinates $q^A = (x^\mu, p_\mu)$, defined to be the coordinates that the symplectic form and Poisson bracket take the form $\omega = dp_\mu \wedge dx^\mu$ and $\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\mu}$ respectively.

Non-Uniqueness of D: The conditions in (3.4) do not uniquely specify D (hence $\Gamma^A{}_{BC}$). It is a fact that we can add an arbitrary tensor Δ_{ABC} symmetric in (ABC) to D and the new connection, let's call it D_{new}:

$$D_{new} \otimes \Theta^A = \Gamma^A{}_{BC}\Theta^C \otimes \Theta^B + \Delta^A{}_{BC}\Theta^C \otimes \Theta^B$$

is a legitimate phase-space connection satisfying same conditions as D, i.e. the conditions in (3.4).

Since D is not unique a choice must be made to start the algorithm. In this thesis D will be the 'cotangent lift' (which fixes D and the symbols $\Gamma^A{}_{BC}$) of the Levi-Civita connection ∇ on the configuration space M equipped with some metric g. We note here that the choice of D in general need not be a cotangent lift of a Levi-Civita connection.

Convention: When we write D^2 we will always mean $D \wedge D$.

3.3.2 Step 2: Fibering the Weyl-Heisenberg Bundle Over Phase-Space

Let (N, ω) be a $2n$-dimensional symplectic manifold. At each point $q \in N$ we introduce a Lie algebra valued covector fiber V_q over N that takes value in some $(2n+1)$ dimensional (Heisenberg) Lie algebra[2]. Let $\hat{v}, \hat{u} \in V_q$, we define this Lie algebra by:

$$[\hat{v}, \hat{u}] = \hat{v}\hat{u} - \hat{u}\hat{v} = i\hbar v_A u_B \omega^{AB} \hat{1}$$

where $\hat{1}$ is the identity in the algebra, $\hat{v} = v_A \hat{y}^A + v_0 \hat{1}$ and $\hat{u} = u_A \hat{y}^A + v_0 \hat{1}$ for some basis $\left(\hat{y}^A, \hat{1}\right)$ for V_q, ω^{AB} are components of the Poisson tensor given by a basis Θ^A of T^*N.

Additionally, we assume there exist a Hermitian conjugation, denoted by †, on V_q, i.e. a map from V_q to V_q such that for any $\hat{v}_1, \hat{v}_2 \in V_q$ and $c_1, c_2 \in \mathbb{C}$:

$$(c_1 \hat{v}_1 + c_2 \hat{v}_2)^\dagger = \bar{c}_1 \hat{v}_1 + \bar{c}_2 \hat{v}_2$$

where the bar (e.g. $\bar{c}$) denotes complex conjugation.

All covectors at $q \in N$, $v = v_A \Theta^A \in T^*N$, along with scalar quantities v_0 (i.e., tensors of rank $\binom{0}{0}$) have a corresponding element $\hat{v} = v_A \hat{y}^A + v_0 \hat{1}$ in V_q. Therefore, these v_A's are really covectors in the cotangent space at q, and these v_0's are really scalar quantities on T^*N. If we require that:

$$D\hat{y}^A = \Gamma^A{}_{BC} \Theta^C \hat{y}^B \quad , \quad D\hat{1} = 0$$

and if (3.3) holds, then all $\hat{v}$'s are coordinate-free objects on N. Additionally, we allow V_q to carry Hilbert space representations at each point $q \in N$, whereby any element of V_q can be represented by linear Hilbert space operators.

Def. The bundle of the algebras V_q, called V, the is defined as $V = \bigcup_{q \in N} V_q$. A point in this bundle is given by the pair $(q, \hat{v})$.

[2] The reason the Lie algebra is $(2n+1)$-dimensional is because $\hat{y}^A$ and the identity element $\hat{1}$ are the $(2n+1)$ basis elements.

Let $\hat{y}^A$ be a basis of V. To better understand these $\hat{y}^A$ (and these $\hat{v} \in V$) rather than as abstract elements in Lie algebra covector bundle over N, we could think of them as a matrix with matrix-elements which are functions of the coordinates $q \in N$. Explicitly we have:

$$\hat{y}^A = \begin{pmatrix} y_{11}^A(q) & y_{12}^A(q) & \cdots \\ y_{21}^A(q) & y_{22}^A(q) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

so that $y_{ij}^A(q) \in C^\infty(N)$ for each i and j. It is assumed that the q's and dq^A's are treated as a scalar with respect to $\hat{y}$'s matrix indices, i.e., $dq^A = \hat{1} dq^A$ (so that $[dq^A, \hat{y}^B] = 0$) and $q^A = \hat{1} q^A$ (so that $[q^A, \hat{y}^B] = 0$). The product of any two elements $\hat{y}^A$ and $\hat{y}^B$ is by definition $\hat{y}^A \hat{y}^B = y_{ij}^A y_{jk}^B = \left(\hat{y}^A \hat{y}^B\right)_{ik}$.

Def. The enveloping algebra of V_q, called the **Weyl-Heisenberg algebra** E_q, at each point $q \in N$ is defined as:

$$E_q = \left\{ \hat{f}(\hat{v}) : \hat{f}(\hat{v}) = \sum_{(l)} f_{(l)jj_1\cdots j_l} (i\hbar)^j \hat{v}^{j_1} \cdots \hat{v}^{j_l} \ , \ f_{(l)jj_1\cdots j_l} \in \mathbb{C} \ , \ \hat{v} \in V_q \right\}$$

Using the basis $\hat{y}^A$ of the vector space V_q defined at a point $q \in N$, each element of $\hat{f} \in E_q$ is expressible as a Taylor series:

$$\hat{f} := \hat{f}(\hat{y}) = \sum_{(l)} f_{(l)j,A_1\cdots A_l} (i\hbar)^j \hat{y}^{A_1} \cdots \hat{y}^{A_l}$$

where the coefficients $f_{(l)j,A_1\cdots A_l} \in \mathbb{C}$ for each index $j, A_1 \cdots A_l$.

Def. The bundle of these algebras called the **Weyl-Heisenberg bundle** E is defined as:

$$E = \bigcup_{q \in N} E_q = \left\{ \hat{f} : \begin{array}{c} \hat{f}(q, \hat{v}) = \sum_{(l)} f_{(l)jj_1\cdots j_l} (i\hbar)^j \hat{v}^{j_1} \cdots \hat{v}^{j_l} \\ f_{jj_1\cdots j_l} \in C^\infty_{\mathbb{C}}(N), \ \hat{v} \in V \end{array} \right\} \tag{3.5}$$

where $C^\infty_{\mathbb{C}}(N)$ is the set of C^∞ complex-valued functions of q on N.

Using a basis $\hat{y}^A$ of the vector-bundle V, each element of $\hat{f} \in E_q$ is expressible as a Taylor series:

$$\hat{f}\left(q,\hat{y}\right) = \sum_{(l)} f_{(l)j,A_1\cdots A_l}\left(i\hbar\right)^j \hat{y}^{A_1}\cdots\hat{y}^{A_l}$$

where the coefficients $f_{(l)j,A_1\cdots A_l} \in C^\infty\left(N\right)$ for each index $j, A_1\cdots A_l$.

The defining properties of the basis $\hat{y}^A$ subsequently are:

$$\left[\hat{y}^A,\hat{y}^B\right] = \hat{y}^B\hat{y}^A - \hat{y}^B\hat{y}^A = i\hbar\omega^{AB}\hat{1} \tag{3.6}$$

$$D\hat{y}^A = \Gamma^A_{\ BC}\Theta^C\hat{y}^B \tag{3.7}$$

where $\hat{1}$ is the identity matrix and whenever a basis of covectors Θ^A is given and the connection on them is:

$$D\otimes\Theta^A = \Gamma^A_{\ BC}\Theta^C\otimes\Theta^B$$

Note: We will omit the $\hat{1}$ from now on and assume that it is implicitly there.

3.3.3 Step 3: Constructing the Global Bundle Connection

We define a matrix derivative called a, global bundle connection, called $\hat{D}$ which is in the form of the graded commutator[3]:

$$\hat{D} = \left[\hat{Q},\cdot\right]/\left(i\hbar\right)$$

where $\hat{Q} \in E\otimes\Lambda$, i.e., $\hat{Q} = \hat{Q}_A\Theta^A$ where $\hat{Q}_A \in E$ so it is of the form:

$$\hat{Q}_A = \sum_l Q_{j,l,AA_1\cdots A_l}\hbar^j\hat{y}^{A_1}\cdots\hat{y}^{A_l} \in E$$

where $Q_{j,l,AA_1\cdots A_l}$ are tensors components that lie in $C^\infty\left(N\right)$, i.e., functions of q that need to be determined. We need $Q_{j,l,AA_1\cdots A_l}$ because it needs to be a globally defined object.

[3]Graded commutators have the property that $\left[\hat{Q}_A\Theta^A,w\right] = \left[\hat{Q}_A,w\right]\Theta^A = \left(\hat{Q}_Aw - w\hat{Q}_A\right)\Theta^A$ where w is an arbitrary l-form with coefficients $w_{A_1\cdots A_l}$ which are complex-valued functions of the variables x, p and $\hat{y}$.

The coefficients $Q_{j,l,AA_1\cdots A_l}$ are partially determined[4] by the condition:

$$\left(D-\hat{D}\right)^2\hat{y}^A=0 \tag{3.8}$$

In the construction of any quantization map this condition is fundamental. As we will see in section 3.6.2, it is a required condition for the quantization σ^{-1} to be one-to-one. That is it assures us that $\sigma^{-1}(f)\in E$ is consistent when parallel propagating around curves, i.e., that it returns to its original value when the curve is closed. Therefore, it seems that this is the fundamental equation and any solution to it will yield a well defined quantization map σ^{-1}. This is why we diverge from Fedosov's original algorithm because we simply need some solution to this equation and we don't really care about any specific one.

In Fedosov's original formulation, he integrates equation (3.8) for a unique $\hat{D}$ iteratively using techniques in cohomology theory (see Fedosov B. 1996 and Gadella M. *et al* 2005). This constructs the coefficients $Q_{j,l,AA_1\cdots A_l}$ term-by-term in increasing powers of $\hbar$.

For the moment forget about Fedosov's specific way of constructing $\hat{D}$ in these references, all we want is a solution to $\hat{D}$ in equation (3.8) for our case of the cotangent bundle of a class of constant curvature manifolds. The advantage of dealing with a specific example is that we can utilize properties of these manifolds to solve the equation. This allows us to diverge from Fedosov's method because his main consideration was to write $\hat{D}$ down for a completely general symplectic manifold.

In this thesis, we chose ansätze in (5.14) and (4.11). By putting these into the condition in (3.8), we obtain a set of differential equations that we subsequently solve to obtain some $\hat{D}$.

3.3.4 Step 4: Defining the Section in the Bundle

In this step we will assign a unique element $\hat{f}\in E$ to every C^∞ phase-space function $f(q)$. The first thing to do is to narrow the choices in this large algebra E, i.e., the enveloping algebra of q and $\hat{y}$. First we define the quantum algebra $\mathcal{Q}_{D,\hat{D}}\subset E$ which will be the enveloping algebra of the algebra of observables:

[4]Fedosov adds an additional condition that makes his $\hat{D}$ unique from a fixed D being $\hat{d}^{-1}r_0=0$ where $\hat{d}^{-1}$ is what he calls δ^{-1} (an operator used in a de Rham decomposition) and r_0 is the first term in the recursive solution. We regard this choice as being artificial and thus omit it from the thesis.

Def. The **quantum algebra** $\mathcal{Q}_{D,\hat{D}} \subset E$ is defined as the set of solutions to the equation:

$$\left(D - \hat{D}\right) \hat{f} = 0$$

by:

$$\mathcal{Q}_{D,\hat{D}} := \left\{ \hat{f}\left(q, \hat{y}\right) : \left(D - \hat{D}\right) \hat{f} = 0 \quad , \quad \hat{f} \in E \right\} \tag{3.9}$$

Def. The **observable algebra** $\mathcal{A}_{D,\hat{D}} \subset \mathcal{Q}_{D,\hat{D}} \subset E$ is defined as the subalgebra of $\mathcal{Q}_{D,\hat{D}}$ of Hermitian operators:

$$\mathcal{A}_{D,\hat{D}} := \left\{ \hat{f}\left(q, \hat{y}\right) : \hat{f} = \hat{f}^{\dagger} \quad , \quad \hat{f} \in \mathcal{Q}_{D,\hat{D}} \right\} \tag{3.10}$$

Remember that a general element of E is of the form:

$$\hat{f}\left(q, \hat{y}\right) = \sum_{(l)} f_{(l)j,A_1 \cdots A_l} \left(i\hbar\right)^j \hat{y}^{A_1} \cdots \hat{y}^{A_l} \tag{3.11}$$

where $f_{(l)j,A_1 \cdots A_l} \in C^{\infty}_{\mathbb{C}}\left(N\right)$ are complex-valued functions of $q \in N$ for each index $j, A_1, \ldots, A_l$.

In Fedosov's original algorithm, he makes the projection σ unique by requiring that the series is symmetric in the $\hat{y}$'s, i.e., σ is the projection of the symmetrized series. Equivalently, you could just require that the coefficients $f_{(l)j,A_1 \cdots A_l}$ be symmetric in the indices $A_1 \cdots A_l$ to obtain the symmetrized series and therefore, uniqueness. We do not symmetrize the $\hat{y}$'s, we will symmetrize other elements. The resulting series will have coefficients that are different than $f_{(l)j,A_1 \cdots A_l}$.

Def. Formal series in $\hbar$ is a power series in $\hbar$ with coefficients in A denoted by adding $[[\hbar]]$ like $A\left[[\hbar]\right]$.

For example, Given a symplectic manifold $\left(N, \omega\right)$ then we define $C^{\infty}\left(N\right)\left[[\hbar]\right]$ as a formal series in $\hbar$ with coefficients in $C^{\infty}\left(N\right)$. Let $f\left(q\right) \in C^{\infty}\left(N\right)\left[[\hbar]\right]$ then

$$f\left(q\right) = f_j\left(q\right)\ \hbar^j = f_0\left(q\right) + f_1\left(q\right)\ \hbar + f_2\left(q\right)\ \hbar^2 + \cdots$$

where $f_j\left(q\right) \in C^{\infty}\left(N\right)$ for each j.

Def. The Fedosov projection $\sigma_F : E \to C^{\infty}(N)[[\hbar]]$ defined by:

$$\sigma_F(\hat{f}) = \sum_j f_{(0)j}\,(i\hbar)^j$$

Note: Fedosov considers this larger class of functions $C^{\infty}(N)[[\hbar]]$, because he is interested in star-products and quantum mechanics on a generalized classical phase-space N.

To construct the Fedosov quantization map σ_F^{-1}, Fedosov then integrates equation (3.12) for a unique set of coefficients $f_{(l)j,A_1\cdots A_l}$ iteratively *for each* $f(q)$ using, again, techniques in cohomology theory (see Fedosov B. 1996 and Gadella M. *et al* 2005). This constructs $\sigma_F^{-1}(f)$ term-by-term in increasing powers of $\hbar$ starting with $f_{(0)} = f(q)$. Forgetting for the moment about Fedosov's specific way of constructing $\sigma_F^{-1}(f)$ in these references, all we want is a consistent way of assigning a unique operator $\sigma_F^{-1}(f)$ to each phase-space function f. Dealing with a specific example allows us to diverge here since Fedosov's main consideration was to write $\sigma_F^{-1}(f)$ (just like $\hat{D}$) down for a completely general symplectic manifold.

Our Modification of Step 4

In this thesis, we take a slight modification to the above. It is a strategy adopted by canonical quantization: choose a basis of phase-space $q^A = (x^{\mu}, p_{\mu})$, quantize it, and force the resulting elements to be a basis of your quantum and observable algebra. The one caveat is that this won't be able to quantize all $f \in C^{\infty}(N)$ just ones that are analytic, i.e. all $f \in C^{\infty}_A(N) \subset C^{\infty}(N)$. In section 3.7 we briefly discuss how we could extend σ^{-1} to a smooth functions, i.e. all $f \in C^{\infty}(N)$.

Therefore, each $\hat{f} \in \mathcal{A}_{D,\hat{D}}$ associated to each phase-space function $f(q) \in C^{\infty}(N)$ must satisfy the condition:

$$\left(D - \hat{D}\right)\hat{f} = 0 \tag{3.12}$$

where we define σ by:

$$\sigma(\hat{f}) = f_{(0)0} = f(q) \quad , \quad \hat{f} = \hat{f}^{\dagger}$$

In other words, $\hat{f} \in E$ is a matrix-valued function of the form (3.11) where $f_{(l)j,A_1\cdots A_l} \in C^\infty_{\mathbb{C}}(N)$ are complex-valued functions of q for each $j, A_1, \ldots, A_l$ that need to be determined by (3.12).

As we will see in section 3.6.1, the physical origin of equation (3.12) is that it allows us to associate some matrix operators called $(\hat{x}^\mu, \hat{p}_\mu)$ to be identified with the differential operators $(\partial/\partial p_\mu, \partial/\partial x^\mu)$ at every point $q \in N$.

There is some additional freedom for determining the map σ^{-1}. However, any such choice will leave the quantum algebra $\mathcal{Q}_{D,\hat{D}}$ invariant. Once a choice is made, we have a unique operator $\hat{f} := \sigma^{-1}(f)$ associated to all $f \in C^\infty_A(N)$. Subsequently, we have the following properties:

$$\sigma\left(\sigma^{-1}(f)\right) = f \quad , \quad \sigma^{-1}\left(\sigma\left(\hat{f}\right)\right) = \hat{f}$$

for any f and $\hat{f}$.

Assume that we have an explicit formula for the (Hermitian) operator $\hat{q}^A := \sigma^{-1}\left(q^A\right) \in \mathcal{A}_{D,\hat{D}}$:

$$\sigma^{-1}\left(q^A\right) = \hat{q}^A(q,\hat{y}) := \sum_{(l)} c^A_{(l)j,A_1\cdots A_l}(i\hbar)^j\,\hat{y}^{A_1}\cdots\hat{y}^{A_l} \in \mathcal{A}_{D,\hat{D}}$$

where $c^A_{(l)j,A_1\cdots A_l} \in C^\infty_{\mathbb{C}}(N)$ for each index $j, A_1\cdots A_l$.

We assume that we can invert this formula to obtain an expression for $\hat{y}^A$ in terms of q^A and $\hat{q}^A$:

$$\hat{y}^A(q,\hat{q}) = \sum_{(l)} b^A_{(l)j,A_1\cdots A_l}(i\hbar)^j\,\hat{q}^{A_1}\cdots\hat{q}^{A_l}$$

where $b^A_{(l)j,A_1\cdots A_l} \in C^\infty_{\mathbb{C}}(N)$ for each index $j, A_1\cdots A_l$.

By having a new basis $\hat{q}^A$ then all elements in $\hat{f} \in E$ can be expressed as a series of the form:

$$\hat{f}(q,\hat{q}) = \sum_{(l)} \tilde{f}_{(l)j,A_1\cdots A_l}(i\hbar)^j\,\hat{q}^{A_1}\cdots\hat{q}^{A_l}$$

where $\tilde{f}_{(l)j,A_1\cdots A_l} \in C^\infty_{\mathbb{C}}(N)$ for each index $j, A_1\cdots A_l$.

By a simple computation, we can see that if $\left(D - \hat{D}\right)\hat{f} = 0$ then each coefficient $\tilde{f}_{(l)A_1\cdots A_l}$ is constant for each index $j, A_1\cdots A_l$. Therefore, any element $\hat{f} \in \mathcal{Q}_{D,\hat{D}}$ has these coefficients being constant.

Thm. If $\hat{y}^A$ can be expressed in terms of q^A and $\hat{q}^A := \sigma^{-1}\left(q^A\right)$ then any element $\hat{f} \in \mathcal{Q}_{D,\hat{D}}$ is expressible as a series of the form:

$$\hat{f}(q,\hat{q}) = \sum_{(l)} \tilde{f}_{(l)j,A_1\cdots A_l} (i\hbar)^j \hat{q}^{A_1}\cdots\hat{q}^{A_l}$$

where $\tilde{f}_{(l)j,A_1\cdots A_l} \in \mathbb{C}$ is constant for each index $j, A_1\cdots A_l$.

Def. We define the (modified) Fedosov quantization map $\sigma^{-1} : C_A^\infty(N) \to \mathcal{A}_{D,\hat{D}}$ by:

$$\begin{aligned} f(q) &= \sum_{(l)} \tilde{f}_{(l)A_1\cdots A_l} q^{A_1}\cdots q^{A_l} \in C_A^\infty(N) \\ \overset{\sigma}{\leftrightarrow} \hat{f}(\hat{q}) &= \sum_{(l)} \tilde{f}_{(l)A_1\cdots A_l} \hat{q}^{A_1}\cdots\hat{q}^{A_l} \in \mathcal{A}_{D,\hat{D}} \end{aligned}$$

where the coefficients $\tilde{f}_{(l)A_1\cdots A_l}$ are constants and are symmetric in the indices $A_1\cdots A_l$.

Note: Fedosov symmetrizes the $\hat{y}$'s and not some basis $\hat{q}^A$ like we have. However, the two different symmetrizations give the same enveloping algebra, i.e., the quantum algebra $\mathcal{Q}_{D,\hat{D}}$ is the same in either case. Also, each $\sigma^{-1}(f)$ is Hermitian iff $\sigma^{-1}\left(q^A\right)$ is Hermitian, i.e., $\sigma^{-1}\left(q^A\right) \in \mathcal{A}_{D,\hat{D}}$ iff $\sigma^{-1}(f) \in \mathcal{A}_{D,\hat{D}}$.

Def. The **analytic observable algebra** $\widetilde{\mathcal{A}}_{D,\hat{D}} \subset \mathcal{A}_{D,\hat{D}}$ is defined as the subalgebra of $\mathcal{A}_{D,\hat{D}}$ determined by the image $\sigma^{-1}\left(C_A^\infty(N)\right)$:

$$\begin{aligned} \widetilde{\mathcal{A}}_{D,\hat{D}} &:= \left\{ \hat{f} : \begin{array}{c} \hat{f}(\hat{q}) = \sum_{(l)} \tilde{f}_{(l)A_1\cdots A_l} \hat{q}^{A_1}\cdots\hat{q}^{A_l} \\ \hat{f} \in \sigma^{-1}\left(C_A^\infty(N)\right) \subset \mathcal{A}_{D,\hat{D}} \\ \tilde{f}_{(l)A_1\cdots A_l} \text{ are symmetric in } A_1\cdots A_l \end{array} \right\} \\ &:= \left\{ \hat{f} : \hat{f} \in \sigma^{-1}\left(C_A^\infty(N)\right) \right\} \end{aligned} \tag{3.13}$$

The algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$ is the symmetrized enveloping algebra of the elements $\hat{q}^A$.

3.3.5 Step 5. The Fedosov Star-Product (Optional)

The Fedosov star-product can now be defined:

Def. The Fedosov star-product $f * g$ is defined by:

$$f *_F g := \sigma(\sigma^{-1}(f)\,\sigma^{-1}(g)) = \sigma(\hat{f}\hat{g}) \tag{3.14}$$

On a technical note, to get the leading order term $\sigma(\hat{f}\hat{g})$ you have to symmetrize all the monomials in $\hat{y}$'s or $\hat{q}$'s in the product $\hat{f}\hat{g}$ first, then take the leading term. This makes the multiplication of $f *_F g$ highly non-trivial.

3.4 QUANTIZING A HAMILTONIAN

In this section we illustrate how to use the map σ^{-1} on a Hamiltonian H and finding the spectrum of H, assuming that σ^{-1} is constructed. By Hamiltonian we mean any function H that generates a Hamiltonian vector-field $\hat{H}$, which then gives the integral curves on the level surfaces of H ($H =$ constant) associated to the dynamics of a physical system.

The Fedosov quantization map σ^{-1} is first used to construct the algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$ as well as a basis $(\hat{x}^\mu, \hat{p}_\mu)$ of $\widetilde{\mathcal{A}}_{D,\hat{D}}$. Second we represent the basis $(\hat{x}^\mu, \hat{p}_\mu)$ by differential operators. This is completely analogous to the replacement $(\hat{x}^\mu, \hat{p}_\mu) \to (x^\mu, -i\hbar\partial/\partial x^\mu)$ in nonrelativistic quantum mechanics in Euclidean space $\mathbb{R}^3$ equipped with the metric δ. In the third step, we write the quantized Hamiltonian for geodesic motion by replacement of $(\hat{x}^\mu, \hat{p}_\mu)$ by (x^μ, T_μ) in the Hamiltonian where $T_\mu(x)$ is a linear differential operator of x^μ:

1. Consider some choice of coordinates $q^A = (x^\mu, p_\mu)$ and the map σ^{-1} on them and define $\hat{x}$ and $\hat{p}$ by:

$$\hat{x}^\mu(x, p, \hat{y}) := \sigma^{-1}(x^\mu) \;\in \mathcal{A}_{D,\hat{D}} \quad , \quad \hat{p}_\mu(x, p, \hat{y}) := \sigma^{-1}(p_\mu) \in \mathcal{A}_{D,\hat{D}}$$

The elements $\hat{x}^\mu$ and $\hat{p}_\mu$ can serve as a basis of our algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$. Therefore, we will express all elements within $\widetilde{\mathcal{A}}_{D,\hat{D}}$ to be matrix-valued functions of these variables.

2. By computing the commutation relations[5]:

$$[\hat{x}^{\mu},\hat{p}_{\nu}]=? \quad , \quad [\hat{p}_{\mu},\hat{p}_{\nu}]=? \quad , \quad [\hat{x}^{\mu},\hat{x}^{\mu}]=0$$

we can represent $\hat{p}_{\nu}$ by some linear differential operator T_{μ} of a variable x^{μ}:

$$T_{\mu}:=T_{\mu}^{\nu}\left(x\right)\frac{\partial}{\partial x^{\nu}}+L_{\mu}\left(x\right)$$

$$\hat{q}^{A}=\left(\hat{x}^{\mu},\hat{p}_{\mu}\right)\rightarrow\hat{Q}^{A}=\left(x^{\mu},-i\hbar T_{\mu}\right)$$

defined in the same way as $(\hat{x},\hat{p})\rightarrow(x,-i\hbar\partial_{x})$ in ordinary quantum mechanics. The justification for this replacement is in section 3.6.1.

3. First we need some Hamiltonian, for example the Hamiltonian of geodesic motion given a metric $g_{\mu\nu}\left(x\right)$ on the configuration space, an arbitrary constant $\xi\in\mathbb{R}$, and the Ricci scalar curvature $R\left(x\right)$ associated to the Levi-Civita connection that preserves $g_{\mu\nu}$:

$$H=g^{\mu\nu}\left(x\right)p_{\mu}p_{\nu}-m^{2}+\xi R\left(x\right) \tag{3.15}$$

We can quantize it using the map σ^{-1}:

$$\hat{H}\left(\hat{x},\hat{p}\right):=\sigma^{-1}\left(H\right)\in E$$

We then let this be a constraint on the set of allowed physical states $|\psi\rangle$ as:

$$\sigma^{-1}\left(H\right)|\phi\rangle=0$$

In x-space (by taking the scalar product with $\langle x|$) we can rewrite the above equation as:

$$\hat{H}\left(x,-i\hbar T\right)\phi\left(x\right)=0$$

where $\hat{H}\left(\hat{x},\hat{p}\right)\rightarrow\hat{H}\left(x,-i\hbar T\right)$ is a differential operator of the variables x^{μ} and dependent on the other differential operator T_{μ}. For example, if $\hat{H}\left(\hat{x},\hat{p}\right)=g^{\mu\nu}\left(\hat{x}\right)\hat{p}_{\mu}\hat{p}_{\nu}-m^{2}+\xi R\left(\hat{x}\right)$ then:

$$\hat{H}\left(x,T\right)=g^{\mu\nu}\left(x\right)T_{\mu}T_{\nu}-m^{2}+\xi R\left(x\right)$$

[5] The x with x commutator will always be zero for a cotangent bundle as can be seen by the Fedosov algorithm.

3.5 THE FEDOSOV STAR-PRODUCT ON $T^*\mathbb{R}^N$

We want to first give a simple example of how the Fedosov algorithm works in the simplest case: the phase-space $T^*\mathbb{R}^n$. This is to give the reader a basic idea of how the algorithm works.

To start in **step 1** of the algorithm, we need the symplectic form ω and some phase-space connection D. The phase-space $T^*\mathbb{R}^n$ has global Darboux coordinates called $q^A = (x^\mu, p_\mu)$ where $\omega = dp_\mu dx^\mu$. The phase-space connection we choose is:

$$Dq^A := dq^A$$

$$D \otimes dq^A := 0$$

Because our phase-space is topologically $T^*\mathbb{R}^n$, we can make such a definition, i.e., where all of the $\Gamma^A{}_{BC}$'s are zero. This completes **step 1**.

Moving to **step 2** we introduce our matrix fibering, i.e., our Weyl-Heisenberg algebra E_q over the phase-space at each point $q \in T^*\mathbb{R}^n$. The bundle of these algebras is the Weyl-Heisenberg bundle E and let $\left(q^A, \hat{y}^A\right)$ be a basis of V. The conditions on the matrix basis $\hat{y}$ are:

$$\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB}$$

$$D\hat{y}^A = 0 \qquad \left(\Gamma^A{}_{BC} = 0\right)$$

Remember that each element of this matrix can be thought as an element in $C^\infty\left(T^*\mathbb{R}^n\right)$.

In **step 3** we must find a $\hat{D}$ that satisfies the condition in (3.8):

$$\left(D - \hat{D}\right)^2 \hat{y}^A = 0$$

where $\hat{D} := \left[\hat{Q}, \cdot\right]$ and $\hat{Q} \in E \otimes \Lambda$ (i.e., $\hat{Q} = \hat{Q}_A \Theta^A$ and $\hat{Q}_A \in E$).

In appendix F we rewrite the condition in (3.8) by some algebraic manipulation to:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = \left[\Omega - D\hat{Q} + \hat{Q}^2/\left(i\hbar\right), \hat{y}^A\right]/\left(i\hbar\right) = 0 \tag{3.16}$$

where $\hat{Q} = \omega_{AB}\hat{y}^A\Theta^B + r$. The term $[\Omega, \cdot]$ is the curvature as a commutator where:

$$\frac{1}{i\hbar}\left[\Omega, \hat{y}^A\right] := D^2\hat{y}^A = R_{CEB}{}^A\Theta^C \wedge \Theta^E \hat{y}^B \tag{3.17}$$

with solution $\Omega := -\frac{1}{2}\omega_{AE}R_{CEB}{}^{A}\Theta^{C}\wedge\Theta^{E}\hat{y}^{B}\hat{y}^{E}$ where $R_{CEB}{}^{A}$ is the phase-space curvature and Θ^{A} is some basis of one-forms.

Any $\hat{Q}$ that satisfies the condition[6]:

$$\Omega - D\hat{Q} + \hat{Q}^{2}/\left(i\hbar\right) = 0 \tag{3.18}$$

also solves the condition in (3.16) (which is the same as (3.8)). We note that we could add something that commutes with all $\hat{y}$'s to $\Omega - D\hat{Q} + \hat{Q}^{2}/\left(i\hbar\right)$ and the condition in (3.16) would still hold.

For our case of $T^{*}\mathbb{R}^{n}$, the Riemann tensor is $R_{CEB}{}^{A} = 0$ therefore the condition (3.18) becomes the simplified condition:

$$-D\hat{Q} + \hat{Q}^{2}/\left(i\hbar\right) = 0 \tag{3.19}$$

By a simple computation, it is verified that a solution for $\hat{Q}$ is:

$$\hat{Q} = \left(x^{\mu} + s^{\mu}\right)dp_{\mu} - \left(p_{\mu} + k_{\mu}\right)dx^{\mu}$$

where $\hat{y}^{A} = \left(s^{\mu}, k_{\mu}\right)$, i.e., s^{μ} are defined to be the first n $\hat{y}$'s and k_{μ} are defined as the last n $\hat{y}$'s with:

$$\left[s^{\mu}, k_{\nu}\right] = i\hbar\delta^{\mu}_{\nu} \quad , \quad \left[s^{\mu}, s^{\nu}\right] = \left[k_{\mu}, k_{\nu}\right] = 0$$

coming from the formulas (3.6) and $\omega = dp_{\mu}dx^{\nu}$.

Here are the computations:

$$D\left(\left(x^{\mu} + s^{\mu}\right)dp_{\mu} - \left(p_{\mu} + k_{\mu}\right)dx^{\mu}\right) = -2dp_{\mu}dx^{\mu}$$

$$\hat{Q}^{2} = \left[\hat{Q}_{A}, \hat{Q}_{B}\right]dq^{A}dq^{B} = -2i\hbar dp_{\mu}dx^{\mu}$$

So that:

$$\hat{D} = \left[\left(x^{\mu} + s^{\mu}\right)dp_{\mu} - \left(p_{\mu} + k_{\mu}\right)dx^{\mu}, \cdot\right]/\left(i\hbar\right)$$

In **step 4** we want to define the analytic observable algebra $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C^{\infty}_{A}\left(T^{*}\mathbb{R}^{n}\right)\right)$ defined in (3.13). Remember that this is the image σ^{-1} applied to all analytic phase-space functions, so all we need is some basis $\sigma^{-1}\left(x^{\mu}\right)$ and $\sigma^{-1}\left(p_{\mu}\right)$.

[6]This is the same as the condition of Fedosov $\Omega - Dr + \hat{d}r + r^{2} = 0$. See Fedosov B. 1996, and Gadella M. *et al* 2005.

Explicitly we want some operators $\hat{x}^\mu, \hat{p}_\nu \in \mathcal{A}_{D,\hat{D}}. \subset E$ (which will be our basis $\sigma^{-1}(x^\mu)$ and $\sigma^{-1}(p_\mu)$) such that the following conditions hold:

$$\left(D - \hat{D}\right)\hat{x}^\mu = 0 \quad , \quad \sigma(\hat{x}^\mu) = x^\mu \tag{3.20}$$

$$\left(D - \hat{D}\right)\hat{p}_\mu = 0 \quad , \quad \sigma(\hat{p}_\mu) = p_\mu$$

Remember that these conditions $\left(D - \hat{D}\right)\hat{x}^\mu = 0$ and $\left(D - \hat{D}\right)\hat{p}_\mu = 0$ are from the definition of $\mathcal{Q}_{D,\hat{D}}$ that contains $\mathcal{A}_{D,\hat{D}}$.

By a very straightforward calculation, we can verify that:

$$\hat{x}^\mu := x^\mu + s^\mu \quad , \quad \hat{p}_\mu := p_\mu + k_\mu$$

satisfy the conditions in (3.20). Therefore, we define:

$$\sigma^{-1}(x^\mu) := \hat{x}^\mu := x^\mu + s^\mu \quad , \quad \sigma^{-1}(p_\mu) := \hat{p}_\mu := p_\mu + k_\mu \tag{3.21}$$

Note: Except for the conditions $\left(D - \hat{D}\right)\hat{x}^\mu = 0$ and $\left(D - \hat{D}\right)\hat{p}_\mu = 0$, the only other requirement is that the term that has no $\hbar$'s and $\hat{y}$'s ($\hat{y}^A = (s^\mu, k_\mu)$) is $\sigma(\hat{x}^\mu) = x^\mu$ for $\sigma^{-1}(x^\mu)$ and $\sigma(\hat{p}_\mu) = p_\mu$ for $\sigma^{-1}(p_\mu)$. The different choices will amount to a choice of ordering, for example standard ordering ($s^\mu k_\nu$) and symmetric Weyl ordering ($\frac{1}{2}(s^\mu k_\nu + k_\nu s^\mu)$).

The commutators of $\sigma^{-1}(x^\mu)$ and $\sigma^{-1}(p_\mu)$, using the formulas in (3.21), are computed to be:

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu \quad , \quad [\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}_\mu, \hat{p}_\nu] = 0$$

Here we see that anything expressible in terms of $\hat{y}$ can be expressed in terms of x, p, $\hat{x}$ and $\hat{p}$ by the simple substitution:

$$\hat{y}^A = (\hat{x}^\mu - x^\mu, \hat{p}_\mu - p_\mu)$$

Now that we have a basis $(\hat{x}^\mu, \hat{p}_\mu)$ for $\widetilde{\mathcal{A}}_{D,\hat{D}}$ then any element $\hat{f} \in \widetilde{\mathcal{A}}_{D,\hat{D}}$ can be written as:

$$\hat{f}(\hat{x}, \hat{p}) = \sum_{(l)(m)} \tilde{f}^{\nu_1 \cdots \nu_m}_{(l)(m)j,\mu_1 \cdots \mu_l} \hbar^j SYM\left(\hat{x}^{\mu_1} \cdots \hat{x}^{\mu_l} \hat{p}_{\nu_1} \cdots \hat{p}_{\nu_m}\right)$$

where:

$$SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\right) = \frac{1}{(l+m)!}\left(\begin{array}{c}\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\\ +\,(\text{all perms. of } \hat{x}\text{'s and } \hat{p}\text{'s})\end{array}\right) \tag{3.22}$$

$$= \frac{1}{(l+m)!}\left(\begin{array}{c}\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\\ +\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_{l-1}}\hat{p}_{\nu_1}\hat{x}^{\mu_l}\hat{p}_{\nu_2}\cdots\hat{p}_{\nu_m}+\cdots\end{array}\right)$$

Therefore, given any $f \in C_A^\infty\left(T^*\mathbb{R}^n\right)$ we have we have a unique operator $\sigma^{-1}(f)$:

$$\begin{aligned} f(x,p) &= \sum_{(l)(m)} \tilde{f}^{\nu_1\cdots\nu_m}_{(l)(m)\mu_1\cdots\mu_l} x^{\mu_1}\cdots x^{\mu_l} p_{\nu_1}\cdots p_{\nu_m} \\ \overset{\sigma}{\leftrightarrow}\ \hat{f}(\hat{x},\hat{p}) &= \sum_{(l)(m)} \tilde{f}^{\nu_1\cdots\nu_m}_{(l)(m)\mu_1\cdots\mu_l} SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\right)\end{aligned}$$

so $\widetilde{\mathcal{A}}_{D,\hat{D}} = \left\{\hat{f} : \hat{f} \in \sigma^{-1}\left(C_A^\infty\left(T^*\mathbb{R}^n\right)\right)\right\}$.

The above formula tells us that $\sigma^{-1} = \mathcal{W}$, the Weyl quantization map (symmetric quantization) and the resulting product will be the Groenewold-Moyal star-product.

Since the commutators are:

$$\left[\hat{x}^\mu, \hat{p}_\nu\right] = i\hbar\delta^\mu_\nu \quad , \quad \left[\hat{x}^\mu, \hat{x}^\nu\right] = \left[\hat{p}_\mu, \hat{p}_\nu\right] = 0$$

we can represent the elements $(\hat{x}^\mu, \hat{p}_\mu)$ by $(x^\mu, -i\hbar\partial/\partial x^\mu)$. Remember that this substitution is justified in section 3.4.

Letting $H = g^{\mu\nu}p_\mu p_\nu - m^2$ where $g^{\mu\nu}$ has constant components, we can write the constraint on the set of allowed physical states $|\psi\rangle$:

$$\sigma^{-1}(H)\,|\psi\rangle = 0$$

i.e., as:

$$\left(g^{\mu\nu}\hat{p}_\mu\hat{p}_\nu - m^2\right)|\psi\rangle = 0$$

In x-space (by taking the scalar product with $\langle x|$) we can rewrite the above equation as:

$$\left(\hbar^2 g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + m^2\right)\psi(x) = 0$$

If the dimension is four and $g^{\mu\nu}$ is Minkowskian, then the above equation is the standard Klein-Gordon equation in Minkowski space.

3.6 THE PHYSICAL ORIGINS OF THE CONDITIONS (3.12) AND (3.8)

In this section we demonstrate the physical meaning of the two most important conditions in the algorithm, equations (3.12) and (3.8):

$$\left(D-\hat{D}\right)^2\hat{y}^A=0 \tag{3.23}$$

$$\left(D-\hat{D}\right)\hat{f}\left(x,p,\hat{y}\right)=0 \tag{3.24}$$

Remember that the quantum algebra $\mathcal{Q}_{D,\hat{D}}$ defined in (3.9) used the condition (3.12) as its main condition for its definition.

3.6.1 The Physical Origin of the Condition in (3.12)

It is important to have a global bundle connection $\hat{D}$ is that well-defined at every point in our symplectic manifold $q \in N$, because the condition (3.12) identifies the operators (that we call the "quantized q^A" or $\hat{q}^A$), which perform infinitesimal translations in all directions. In other words, the action of these operators $\left[\hat{q}^A,\cdot\right]$ translates other observables from one point $q \in N$ to some other point infinitesimally close to q. Once these infinitesimal operators have been constructed, finite translations may be obtain using standard methods.

To justify the above statement we summarize the example of $T^*\mathbb{R}^n$ done in section 3.5 to clarify what we mean:

E.g. For $T^*\mathbb{R}^n$ a solution for $\hat{D}$ was computed to be:

$$\hat{D}=\left[\left(x^\mu+s^\mu\right)dp_\mu-\left(p_\mu+k_\mu\right)dx^\mu,\cdot\right]/\left(i\hbar\right)$$

then the equation in (3.24) reduces to:

$$\frac{\partial\hat{f}}{\partial x^\mu}dx^\mu+\frac{\partial\hat{f}}{\partial p_\mu}dp_\mu=-\frac{1}{i\hbar}\left[\left(p_\mu+k_\mu\right),\hat{f}\right]dx^\mu+\frac{1}{i\hbar}\left[\left(x^\mu+s^\mu\right),\hat{f}\right]dp_\mu$$

This means that $-\frac{1}{i\hbar}\left[\left(p_\mu+k_\mu\right),\cdot\right]$ and $\frac{1}{i\hbar}\left[\left(x^\mu+s^\mu\right),\cdot\right]$ act as $\frac{\partial}{\partial x^\mu}$ (an x-translation) and $\frac{\partial}{\partial p_\mu}$ (a p-translation) on the set of solutions to (3.24), i.e., for all $\hat{f}\in\mathcal{Q}_{D,\hat{D}}$. Therefore, we think of $-\frac{1}{i\hbar}\left[\left(p_\mu + k_\mu\right), \cdot\right]$ and $\frac{1}{i\hbar}\left[\left(x^\mu + s^\mu\right), \cdot\right]$ as the matrix analogue of x and p translations.

This is the fundamental step in quantization, associating a seeming unconnected matrix operator such as $-\frac{1}{i\hbar}\left[\left(p_\mu + k_\mu\right), \cdot\right]$ and give it some physical meaning by knowing its action does a real and physical operation such as an x-translation on the set $\mathcal{Q}_{D,\hat{D}}$. This is what justifies a replacement of these operators and derivatives on the coordinates:

$$-\frac{1}{i\hbar}\left[\left(p_\mu + k_\mu\right), \cdot\right] \to \partial/\partial x^\mu \quad , \quad \frac{1}{i\hbar}\left[\left(x^\mu + s^\mu\right), \cdot\right] \to \partial/\partial p_\mu$$

The notation that we use for these operators under this association is $\hat{p}_\mu$ and $\hat{x}^\mu$:

$$-\frac{1}{i\hbar}\left[\hat{p}_\mu, \cdot\right] \to \partial/\partial x^\mu \quad , \quad \frac{1}{i\hbar}\left[\hat{x}^\mu, \cdot\right] \to \partial/\partial p_\mu \tag{3.25}$$

This notation reflects that in classical mechanics it is known that $\partial/\partial x^\mu = -\left[p_\mu, \cdot\right]_P$ and $\partial/\partial p_\mu = \left[x^\mu, \cdot\right]_P$ where $[,]_P$ is the Poisson bracket.

Also, $\hat{q}^A = (\hat{x}^\mu, \hat{p}_\mu)$ is an observable so we have the condition that $\hat{q}^A \in \mathcal{Q}_{D,\hat{D}}$.

A general symplectic manifold (N, ω) is locally isomorphic to $\mathbb{R}^{2n}$. Therefore, existence of a bundle connection $\hat{D}$ defined globally is a guarantee that at every point $q \in N$ we can make an analogous association. That is we can locally always identify our q-translation operators $\hat{q}^A = (\hat{x}^\mu, \hat{p}_\mu)$ in some Darboux coordinates $q^A = (x^\mu, p_\mu)$.

On general symplectic manifold (N, ω) the procedure is as follows:

1. Every symplectic manifold N of dimension $2n$ is locally, i.e. at every point $q \in N$, isomorphic to $\mathbb{R}^{2n}$ in Darboux coordinates $q^A = (x^\mu, p_\mu)$ where $\omega = dp_\mu dx^\mu$. Specifically, this means that in these Darboux coordinates $q^A = (x^\mu, p_\mu)$ the symbols $\Gamma^A{}_{BC}$ defined in (3.3) are all zero:
$$Df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial p_\mu} dp_\mu$$
$$D \otimes dx^\mu = 0 \quad , \quad D \otimes dp_\mu = 0$$

2. If there exists a global bundle connection $\hat{D} = \left[\hat{Q}_A dq^A, \cdot\right]$ that satisfies (3.24), then equation (3.24) becomes:

$$\frac{\partial \hat{f}}{\partial x^\mu} dx^\mu + \frac{\partial \hat{f}}{\partial p_\mu} dp_\mu = \frac{1}{i\hbar} \left[\hat{F}^\mu dp_\mu - \hat{G}_\mu dx^\mu, \hat{f}\right]$$

where $\hat{F}^\mu$ and $\hat{G}_\mu$ are defined to be the components of $\hat{Q}_A$, i.e., $\hat{Q}_A = \left(\hat{F}^\mu, \hat{G}_\mu\right)$.

This equation means that on the set of solutions to (3.24), defined as $\mathcal{Q}_{D,\hat{D}}$, to each function $\hat{f} \in \mathcal{Q}_{D,\hat{D}}$ the operation of $\frac{1}{i\hbar}\left[\hat{F}^\mu, \cdot\right]$ and $\frac{1}{i\hbar}\left[\hat{G}_\mu, \cdot\right]$ on $\hat{f}$ is identical to $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial p_\mu}$ respectively. These matrix operators $\frac{1}{i\hbar}\left[\hat{F}^\mu, \cdot\right]$ and $\frac{1}{i\hbar}\left[\hat{G}_\mu, \cdot\right]$ now have a connection to real physical operations—x and p (or just q) translations on N.

This is what justifies a replacement of these operators and derivatives on the coordinates:

$$\frac{1}{i\hbar}\left[\hat{Q}_A, \cdot\right] \to D_A$$

The notation that we use for these operators under this association is $\hat{q}^A = (\hat{x}^\mu, \hat{p}_\mu)$. However, this won't uniquely specify $\hat{q}^A$ since we could add a term to each that commute with everything like an arbitrary function of q, but before we show how this freedom is fixed we want to define the algebra.

3. It makes sense to define the quantum algebra as in (3.9):

$$\mathcal{Q}_{D,\hat{D}} := \left\{\hat{f}\left(x, p, \hat{y}\right); \left(D - \hat{D}\right)\hat{f} = 0 \quad , \quad \hat{f} \in E\right\} \tag{3.26}$$

because it is this equation $\left(D - \hat{D}\right)\hat{f} = 0$ which gives a real physical meaning to these sets of operators.

4. As we noted above, we could add to $\hat{Q}_A$ an arbitrary function $f_A(q)$ and $\frac{1}{i\hbar}\left[\hat{Q}_A + f_A, \cdot\right] = \frac{1}{i\hbar}\left[\hat{Q}_A, \cdot\right]$. To fix this freedom we require that $\hat{Q}_A$ be an element in the quantum algebra $\hat{Q}_A \in \mathcal{Q}_{D,\hat{D}}$. This is entirely natural since the meaning of the set of observables is the set of allowable transformations on states and certainly translations on N fall under this category. We then can state that:

$$\hat{q}^A := \hat{Q}^A$$

3.6.2 The Physical Origin of Condition in (3.8)

Another important issue is one of consistency in the Fedosov quantization map σ^{-1}. Remember that σ^{-1} assigns a unique operator $\sigma^{-1}(f)$ to every $f \in C^{\infty}(N)$, i.e., σ^{-1} is one-to-one. This means that if move our operator $\sigma^{-1}(f)$ around any closed curve γ in a contractible domain in a symplectic manifold (N, ω) (i.e., no holes) we'd better get the same operator as before $\sigma^{-1}(f)$ or else σ^{-1} not a good quantization map.

How this relates the condition in (3.23) is actually quite simple knowing a theorem provided in Fedosov B. 1996 in Theorem 1.2.6–2 on p. 23):

Thm. 1.2.6 Let V be a contractible domain in a symplectic manifold (N, ω), and let τ be a parallel transport along a closed curve $\gamma \subset V$ of the algebra E defined in (3.5). Then the equation:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = 0 \tag{3.27}$$

is satisfied iff $\tau_\gamma = id$ for any γ.[7]

This means that if we parallel transport an element $\hat{f} \in \mathcal{Q}_{D,\hat{D}}$ around some any closed curve γ, in general, the new element $\tau_\gamma \hat{f}$ without the condition that $\tau_\gamma = id$, $\hat{f}$, in general, will be different from $\tau_\gamma \hat{f}$, where we call this new operator $\hat{h} = \tau_\gamma \hat{f}$. This causes inconsistencies in the map σ^{-1} because parallel transporting any function $f \in C^{\infty}(N)$ around a closed loop is, by definition, trivial, i.e., $f(\gamma(q_0)) = f(q_0)$. So on the one hand $\sigma^{-1}(f) = \hat{f}$ and on the other (by parallel transport) $\sigma^{-1}(f) = \hat{h}$, violating the one-to-one nature of σ^{-1}. Therefore:

$$\tau_\gamma \circ \sigma^{-1}(f) = \sigma^{-1}(f)$$

for all closed curves $\gamma \subset V$ and since $f \in C^{\infty}(N)$ is arbitrary then $\tau_\gamma = id$, which then implies that (3.27) must hold by Theorem 1.2.6 above.

[7] Fedosov calls any $\hat{D}$ that satisfies the condition $\left(D - \hat{D}\right)^2 \hat{y}^A = 0$ abelian.

3.7 THE FEDOSOV STAR-PRODUCT ON SMOOTH FUNCTIONS

On a technical note, we want to illustrate how the Fedosov star-product is valid for all C^∞ (smooth) functions, not just for functions expressible as a Taylor series. So far we have demonstrated the correspondence:

$$\begin{aligned} f(x,p) &= \sum_{(l)(m)} \tilde{f}^{\nu_1\cdots\nu_m}_{(l)(m)\mu_1\cdots\mu_l} x^{\mu_1}\cdots x^{\mu_l} p_{\nu_1}\cdots p_{\nu_m} \\ \overset{\sigma}{\leftrightarrow} \hat{f}(\hat{x},\hat{p}) &= \sum_{(l)(m)} \tilde{f}^{\nu_1\cdots\nu_m}_{(l)(m)\mu_1\cdots\mu_l} SYM\left(\hat{x}^{\mu_1}\cdots \hat{x}^{\mu_l}\hat{p}_{\nu_1}\cdots\hat{p}_{\nu_m}\right) \end{aligned}$$

where $\hat{x}^\mu$ and $\hat{p}_\mu$ are some Hilbert space operators defined explicitly in terms of the operators $\hat{y}$ as well as the variables x and p.

This is a good start, but this only takes care of the functions $C^\infty_A(N)$. What remains unclear is how to define the action of the map σ^{-1} on smooth functions like the function e^{-1/x^2} or partitions of unity in manifold theory which are not determined everywhere by their Taylor series, i.e., all functions $f \in C^\infty(N)$, but $f \notin C^\infty_A(N)$. This can be explained by knowing that the Groenewold-Moyal star-product is defined on all smooth functions and not just Taylor series including functions like e^{-1/x^2}. In Fedosov's original construction of the star-product, he used a bundle over the phase-space, called the Weyl bundle, of ordinary covectors that has a Groenewold-Moyal-like product between each covector (see Gadella M. *et al* 2005). This is different than the bundle we used, which was a matrix-valued bundle.

Products of two arbitrary operators $f(x,p,y)$ and $g(x,p,y)$ are:

$$f(y)\circ g(y) = \sum_{(j)}^{\infty} (i\hbar/2)^j\, \omega^{A_1B_1}\cdots\omega^{A_jB_j}/j!(\hat{\partial}_{A_1}\cdots\hat{\partial}_{A_j} f)(\hat{\partial}_{B_1}\cdots\hat{\partial}_{B_j} g)$$

$$\hat{\partial}_B y^A = \delta^A_B \quad , \quad \hat{\partial}_B q^A = 0 \quad , \quad \hat{\partial}_B dq^A = 0 \quad , \quad q^A = (x^\mu, p_\mu) \quad , \quad \left[y^A, y^B\right]_\circ = i\hbar\omega^{AB}$$

In this way we can extend the Fedosov star-product to *all* smooth functions. However, to simplify the discussions and computations in this thesis, we only consider $C^\infty_A(N)$ and not the full set $C^\infty(N)$.

4.0 FEDOSOV QUANTIZATION ON THE TWO-SPHERE

In this chapter we present original results of this thesis, calculating the quantization map σ^{-1} for $\mathbb{S}^2$ as well as analyzing the image $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C^\infty_A\left(T^*\mathbb{S}^2\right)\right)$, i.e., the algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$. This is done by implementing the algorithm in section 3.3.

In section 4.1, we define our phase-space connection as well as explain now the constraints are dealt with which is **step 1** of the algorithm. **Step 2** of the algorithm is in section 4.2 where we define the Weyl-Heisenberg bundle over the phase-space of the two-sphere. In section 4.3 we construct our globally defined bundle connection $\hat{D}$ as prescribed in **step 3** of the algorithm. The globally defined bundle connection $\hat{D}$ is constructed exactly by the choice of a particular ansatz. Finally in section 4.4, we construct the quantization map σ^{-1} and analyze its image $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C^\infty_A\left(T^*\mathbb{S}^2\right)\right)$. Here we see that $\widetilde{\mathcal{A}}_{D,\hat{D}}$ is a constrained version of the Euclidean group with the angular momentum subgroup $\mathbb{SO}\left(3\right)$.

4.1 DEFINING THE PHASE-SPACE CONNECTION

The starting place of the Fedosov algorithm is the symplectic form and the phase-space connection in **step 1** of the algorithm in section 3.3. In this section we are making a choice of a connection on the phase-space of the two-sphere $\mathbb{S}^2$. Note that this choice is not unique, but a choice must be made. We choose a connection that is known as the 'cotangent lift' of the Levi-Civita connection on the configuration space. Before we go into the details of this "lifting", we briefly review the embedding geometry of $\mathbb{S}^2 \subset \mathbb{R}^3$.

We use the coordinates given by embedding $\mathbb{S}^2$ into a 3-dimensional Euclidean space $(\mathbb{R}^3, \delta)$ where δ is the Euclidean metric (what we use to raise and lower lower-case Latin and greek indices). The embedding formulas that we use describe a unit sphere centered at the origin of $(\mathbb{R}^3, \delta)$ is:

$$\underline{x} \cdot \underline{x} = 1 \quad , \quad \underline{x} \cdot \underline{p} = 0 \tag{4.1}$$

where $\underline{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ and $\underline{x} \cdot \underline{y} := \delta_{\mu\nu} x^\mu y^\nu = x^1 y^1 + x^2 y^2 + x^3 y^3$. The second constraint $\underline{x} \cdot \underline{p} = 0$ is a secondary constraint associated to the primary constraint $\underline{x} \cdot \underline{x} = 1$.

The basis of covectors we choose to use are:

$$\underline{\alpha} := \underline{x} \times d\underline{p}$$

$$\underline{\theta} := \underline{x} \times d\underline{x}$$

where $\times$ is the standard cross-product defined by $\underline{a} \times \underline{b} = \varepsilon_{\mu\rho\nu} a^\rho b^\nu$ and $\varepsilon_{\mu\rho\nu}$ is the completely antisymmetric tensor.

The embedding metric δ induces a metric on $\mathbb{S}^2$:

$$g = \underline{\theta} \cdot \underline{\theta} \tag{4.2}$$

and the symplectic form on $\mathbb{S}^2$ and $T^*\mathbb{S}^2$ is:

$$\omega = (\delta^\mu_\nu - C x^\mu x_\nu)\, \alpha_\mu \theta^\nu \tag{4.3}$$

where each index of the coefficients $(\delta^\mu_\nu - C x^\mu x_\nu)$ is projected orthogonal to x because of the embedding.

By introducing a 3-dimensional coordinate patch of the sphere, which is a 2-dimensional manifold, the third coordinate is redundant. The constraint $\underline{x} \cdot \underline{x} = 1$ fixes this redundancy. Because of this constraint, the condition that the connection ∇ is metric g preserving and torsion-free ($\nabla \wedge \nabla \otimes d\underline{x} = 0$) is not enough to fix it in the three coordinates, i.e. in $(\mathbb{R}^3, \delta)$. What we additionally need is all derivatives of the constraint equation vanish:

$$\underbrace{\nabla \otimes \cdots \otimes \nabla}_{l} (\underline{x} \cdot \underline{x} - 1) = 0$$

for all $l \in \mathbb{Z}^+$ ($\mathbb{Z}^+$ is the set positive integers). The first derivative ($l = 1$) of the above equation is the constraint $\underline{x} \cdot d\underline{x} = 0$. Taking this constraint into account, the torsion-free Levi-Civita configuration space connection is computed to be:

$$\nabla \otimes \underline{\theta} = \underline{\theta} \otimes_\times \underline{\theta}$$

The way the cotangent lifting procedure works is that you start with a Levi-Civita connection ∇ on the configuration space. Next we define D acting on $\underline{x}$ and $\underline{p}$ to be $d\underline{x}$ and $d\underline{p}$, respectively. Next we define $D \otimes \underline{\theta} = \nabla \otimes \underline{\theta}$ in this case we have:

$$D \otimes \underline{\theta} = \underline{\theta} \otimes_\times \underline{\theta}$$

To fix $D \otimes d\underline{p}$ we have several conditions to aid us:

1. It must preserve the symplectic form $D \otimes \omega = 0$
2. It must be torsion-free $D \wedge D \otimes d\underline{x} = 0$ (which already holds) and $D \wedge D \otimes d\underline{p} = 0$
3. It preserves the constraints $\underline{x} \cdot \underline{x} = 1$ (which already holds) and $\underline{x} \cdot \underline{p} = 0$, i.e.:

$$\underbrace{D \otimes \cdots \otimes D}_{l} (\underline{x} \cdot \underline{x} - 1) = 0 \tag{4.4}$$

$$\underbrace{D \otimes \cdots \otimes D}_{l} \left(\underline{x} \cdot \underline{p}\right) = 0$$

for all $l \in \mathbb{Z}^+$.

The solution for phase-space connection D that satisfies the conditions is computed to be:

$$D\underline{x} := d\underline{x} = \underline{\theta} \times \underline{x} \tag{4.5}$$

$$D\underline{p} := d\underline{p} = \underline{\alpha} \times \underline{x} - \underline{p} \times \underline{\theta}$$

$$D \otimes \underline{\theta} = \underline{\theta} \otimes_\times \underline{\theta}$$

$$D \otimes \underline{\alpha} = \underline{\theta} \otimes_\times \underline{\alpha} - \frac{2}{3} (\underline{\theta} \times \underline{x}) \otimes \left(\underline{p} \cdot \underline{\theta}\right) + \frac{1}{3} \left(\underline{p} \cdot \underline{\theta}\right) \otimes (\underline{\theta} \times \underline{x})$$

Along with its computed curvature:

$$D^2\underline{x} := 0 \tag{4.6}$$

$$D^2\underline{p} := 0$$

$$D^2 \otimes \underline{\theta} = \tilde{\omega} \otimes (\underline{x} \times \underline{\theta})$$

$$D^2 \otimes \underline{\alpha} = \tilde{\omega} \otimes (\underline{x} \times \underline{\alpha}) + \frac{1}{3}\left(\underline{\alpha}\,(\underline{\theta} \otimes . \,\underline{\theta}) - \underline{\theta}\,(\underline{\alpha} \otimes . \,\underline{\theta}) - 2\omega \otimes \underline{\theta}\right)$$

where $\tilde{\omega} := \underline{x} \cdot (\underline{\theta} \times \underline{\theta})$ is another nondegenerate two form (which makes the two-sphere itself a symplectic manifold).

Since we have constraints due to the embedding in (4.1), as mentioned before, we need all derivatives of the constraints to be zero. This forces us to have constraints on $d\underline{x}$ and $d\underline{p}$ resulting from derivatives of the embedding conditions in equation (4.1):

$$\underline{x} \cdot d\underline{x} = 0 \quad , \quad d\underline{x} \cdot \underline{p} + \underline{x} \cdot d\underline{p} = 0$$

With these constraints holding, all derivatives D of the constraints in (4.1) are zero.

4.2 FIBERING THE WEYL-HEISENBERG BUNDLE OVER PHASE-SPACE

We defined $\hat{y}^A$ in **step 2** of the algorithm in section 3.3 and now we let $\hat{y}^A = (s^\mu, k_\mu)$, i.e., s^μ are the first three $\hat{y}$'s and k_μ are the last three. Using the formula for ω in (4.3) and the definition of the commutators in formula (3.6) the commutators are:

$$[s^\mu, s^\nu] = 0 = [k_\mu, k_\nu] \;\;,\;\; [s^\mu, k_\nu] = i\hbar\,(\delta^\mu_\nu - x^\mu x_\nu)$$

along with the constraints $\delta_{\mu\nu} x^\mu s^\nu = x^\mu k_\mu = 0$.

Additionally, the action of the connection and curvature acting on $\underline{s}$ & $\underline{k}$ is written down directly from the equations (5.7), (5.8), (4.5), (4.6), and (4.2):

$$D\underline{s} = \underline{\theta} \times \underline{s} \tag{4.7}$$

$$D\underline{k} = \underline{\theta} \times \underline{k} - \frac{2}{3}\underline{\theta} \times \underline{x}\left(\underline{p} \cdot \underline{s}\right) + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{s} \times \underline{x}\right)$$

$$D^2\underline{s} = \tilde{\omega}\left(\underline{x} \times \underline{s}\right)$$

$$D^2\underline{k} = \tilde{\omega}\left(\underline{x} \times \underline{k}\right) + \frac{1}{3}\left(\underline{\alpha}\left(\underline{s} \cdot \underline{\theta}\right) + \left(\underline{s} \cdot \underline{\alpha}\right)\underline{\theta} - 2\omega\underline{s}\right)$$

where again $\tilde{\omega} := \underline{x} \cdot \left(\underline{\theta} \times \underline{\theta}\right)$ is another nondegenerate two form (which makes the two-sphere itself a symplectic manifold).

4.3 CONSTRUCTING THE GLOBALLY DEFINED BUNDLE CONNECTION

We are ready to begin computing the globally defined bundle connection $\hat{D}$ by means of the equation (3.8), i.e., **step 2** of the algorithm in section 3.3. As mentioned already, the physical origins of this condition is that the Fedosov quantization map σ^{-1} will be one-to-one iff this equation is satisfied. More explicitly, without the condition in (3.8) the operator $\sigma^{-1}(f)$ associated to $f \in C^{\infty}(T^*M)$ wouldn't be consistently parallely propagated around all closed curves.

We may rewrite the condition (3.8):

$$\left(D - \hat{D}\right)^2 \hat{y}^A = 0$$

in a more convenient form to aid our calculations by:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = \left[\Omega - Dr + \hat{d}r + r^2/\left(i\hbar\right), \hat{y}^A\right]$$

so that:

$$\left[\Omega - Dr + \hat{d}r + r^2/\left(i\hbar\right), \hat{y}^A\right]/\left(i\hbar\right) = 0 \tag{4.8}$$

where:

$$\Omega := -\frac{1}{2}\omega_{AC}R_{CEB}{}^{A}\Theta^C \wedge \Theta^E \hat{y}^B \hat{y}^C$$

$$\hat{D} = \left[\hat{Q}, \cdot\right]/\left(i\hbar\right) = \hat{d} + \left[r, \cdot\right]/\left(i\hbar\right) \quad , \quad \hat{d} = \left[\omega_{AB}\hat{y}^A\Theta^B, \cdot\right]/\left(i\hbar\right)$$

where $\hat{Q} \in E \otimes \Lambda$ and Ω is the phase-space curvature as a commutator:

$$\left[\Omega, \hat{y}^A\right] / (i\hbar) := D^2 \hat{y}^A = R_{CEB}{}^A \Theta^C \wedge \Theta^E \hat{y}^B \tag{4.9}$$

and $\hat{d}$ is the flat phase-space solution to $\hat{D}$ (as can be seen in section 3.5). See appendix F for the proof of equation (4.8)—the equation is derived by means of basic algebraic manipulation.

A solution for r in equation (4.8) will determine the operator $\hat{D}$ because $\hat{D} = \hat{d} + [r, \cdot]$ where $\hat{d} = \left[\omega_{AB} \hat{y}^A \Theta^B, \cdot\right]$. It is obvious that a solution to (4.8) for $\hat{D}$ is determined by any r that satisfies:

$$\Omega - Dr + \hat{d}r + r^2 / (i\hbar) = 0 \tag{4.10}$$

Important: To state clearly what has been done so far is that we have derived that a solution for r in equation (4.10) will give us a solution to $\hat{D} = \left[\omega_{AB} \hat{y}^A \Theta^B + r, \cdot\right]$ which satisfies the condition in (3.8). This is the equation we now solve.

We need to now compute Ω for $T^*\mathbb{S}^2$. Ω is determined by the curvature formulas in (4.7):

$$\Omega := \frac{1}{3}\left(\left(\underline{s} \cdot \underline{\alpha}\right)\left(\underline{s} \cdot \underline{\theta}\right) - s^2 \omega\right) + \left(\underline{x} \times \underline{k}\right) \cdot \underline{s} \tilde{\omega}$$

We can easily verify that Ω gives the curvature as a commutator:

$$\left[\Omega, \underline{s}\right] / (i\hbar) = \left[-\underline{k} \cdot \left(\underline{x} \times \underline{s}\right) \tilde{\omega}, \underline{s}\right] = \tilde{\omega}\left(\underline{x} \times \underline{s}\right)$$

$$\left[\Omega, \underline{k}\right] / (i\hbar) = \frac{1}{3}\left(\underline{\alpha}\left(\underline{s} \cdot \underline{\theta}\right) + \left(\underline{s} \cdot \underline{\alpha}\right) \underline{\theta} - 2\omega \underline{s}\right) + \left(\underline{x} \times \underline{k}\right) \tilde{\omega}$$

which agrees with (4.6) and (4.9).

The formula for $\hat{d}$ for $T^*\mathbb{S}^2$ is:

$$\hat{d} = \left[\omega_{AB} \hat{y}^A \Theta^B, \cdot\right] / (i\hbar) = \left[\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta}, \cdot\right] / (i\hbar)$$

To find a solution for r in equation (4.10) we choose an ansatz and plug it into (4.10). In appendix B.2 a lengthy calculation ensues, which consists as a series of derivatives and commutators, all being straightforward though lengthy. Our particular ansatz we chose is:

$$r = r_0 + f\left(s^2\right) \underline{z} \cdot \underline{s}\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta} + g\left(s^2\right) \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{s} \cdot \underline{\theta} + h\left(s^2\right) \underline{s} \cdot \underline{\theta} \tag{4.11}$$

where $\underline{z} = \underline{p} - \underline{x} \times \underline{k}$ and $r_0 = \frac{1}{3}\left(\left(\underline{k} \cdot \underline{\theta}\right) s^2 - \underline{k} \cdot \underline{s}\left(\underline{s} \cdot \underline{\theta}\right)\right)$.

Given the ansatz above, we can compute each term in the condition (4.10) in a straightforward manner (again see appendix B.2 for the calculations):

$$Dr = \left(\frac{1}{9} - \frac{2g}{3} + \frac{f}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + f\underline{\alpha} \cdot (\underline{x} \times \underline{s})(\underline{x} \times \underline{s}) \cdot \underline{\theta} - g(\underline{s} \cdot \underline{\alpha})\,\underline{s} \cdot \underline{\theta}$$

$$\hat{d}r = -\Omega + \left(2f' s^2 + 3f + g\right) \underline{z} \cdot \underline{s}\tilde{\omega} - g(\underline{s} \cdot \underline{\alpha})\,\underline{s} \cdot \underline{\theta} + f\underline{\alpha} \cdot (\underline{x} \times \underline{s})(\underline{x} \times \underline{s}) \cdot \underline{\theta}$$

$$r^2 = \left(\frac{1}{9} - \frac{2g}{3} + \frac{f}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2gf' s^2 + gf - f^2 - \frac{2f}{3} + \frac{g}{3} - \frac{1}{9}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}$$

where $f' = \frac{\partial f}{\partial (s^2)}$ for all functions.

Putting these into the equation (4.10) we obtain a condition for g:

$$g = \frac{\begin{pmatrix} \left(f + \frac{1}{3}\right)^2 - 2f'hs^2 - 3fh - h \\ +2hh's^2 + 2h^2 - 2f' + 2h' \end{pmatrix} s^2 + 2h - 3f}{\left(f + \frac{1}{3} + 2f's^2 - 2h's^2 - h\right) s^2 + 1} \tag{4.12}$$

while f and h are left arbitrary as long as g is well-defined. This is a necessary and sufficient condition for r, so that the resulting $\hat{D}$ satisfies the condition (3.8).

A nice fact of the messy condition in (4.12) is that it yields constant solutions $\left\{f = -\frac{1}{3}, g = 1, h = 0\right\}$ and $\left\{f = -\frac{1}{12}, g = \frac{1}{4}, h = 0\right\}$. We choose the solution $\left\{f = -\frac{1}{3}, g = 1, h = 0\right\}$ for the sake of clarity because future computations will be made much easier. However, in the general case where, i.e., for all well defined solutions to g, f, and h the commutators of $\sigma^{-1}(x^\mu)$ and $\sigma^{-1}(p_\mu)$ were computed to be the same Lie algebra regardless of the choice of g, f, and h.

The solution for r with $\left\{f = -\frac{1}{3}, g = 1, h = 0\right\}$ simplifies to:

$$r = -\frac{1}{3}(\underline{p} \cdot \underline{s})((\underline{x} \times \underline{s}) \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\,\underline{s} \cdot \underline{\theta} \tag{4.13}$$

$$\hat{Q} = \underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta} - \frac{1}{3}(\underline{p} \cdot \underline{s})((\underline{x} \times \underline{s}) \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\,\underline{s} \cdot \underline{\theta} \tag{4.14}$$

$$\hat{D} = \left[\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta} - \frac{1}{3}(\underline{p} \cdot \underline{s})((\underline{x} \times \underline{s}) \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\,\underline{s} \cdot \underline{\theta}, \cdot\right] \tag{4.15}$$

4.4 DEFINING THE SECTION IN THE BUNDLE: OUR OBSERVABLE ALGEBRA

In this section we will construct the map σ^{-1} as in **step 4** in section 3.3. To reiterate, in this thesis, we only construct $\sigma^{-1}\left(C_A^\infty\left(T^*M\right)\right)$ where $C_A^\infty\left(T^*M\right)$ is the space of analytic functions in this section just as in the case for $T^*\mathbb{R}^n$ in section 3.5. See in section 3.7 how σ^{-1} is extended to all $C^\infty\left(T^*M\right)$.

The construction of $\tilde{\mathcal{Q}}_{D,\hat{D}} = \sigma^{-1}\left(C_A^\infty\left(T^*\mathbb{S}^2\right)\right)$ is straightforward once you have some basis $\sigma^{-1}\left(x^\mu\right)$ and $\sigma^{-1}\left(p_\mu\right)$ and the enveloping algebra is the quantum algebra $\tilde{\mathcal{Q}}_{D,\hat{D}}$. The goal now is to find a suitable definition of these elements. Explicitly we want some elements $\hat{x}^\mu, \hat{p}_\nu \in E$ that satisfies the conditions:

$$\left(D-\hat{D}\right)\hat{x}^\mu = 0 \quad , \quad \sigma\left(\hat{x}^\mu\right) := b^\mu_{(0)0} = x^\mu \tag{4.16}$$

$$\left(D-\hat{D}\right)\hat{p}_\mu = 0 \quad , \quad \sigma\left(\hat{p}_\mu\right) := c_{(0)0,\mu} = p_\mu \tag{4.17}$$

where:

$$\hat{x}^\mu = \sum_{(l)} b^\mu_{(l)j,A_1\cdots A_l}\hbar^j\hat{y}^{A_1}\cdots\hat{y}^{A_l} \tag{4.18}$$

$$\hat{p}_\mu = \sum_{(l)} c_{(l)j\mu,A_1\cdots A_l}\hbar^j\hat{y}^{A_1}\cdots\hat{y}^{A_l} \tag{4.19}$$

Therefore, it is our job to find some solutions for these coefficients $b^\mu_{(l)j,A_1\cdots A_l}$ and $c_{(l)j\mu,A_1\cdots A_l}$ that satisfies the conditions in (4.16) and (4.17). To help us find an exact solution for $T^*\mathbb{S}^2$, we choose the ansätze for $\hat{x}$ and $\hat{p}$ to be:

$$\underline{\hat{x}} = v\left(s^2\right)\underline{x} + w\left(s^2\right)\underline{x}\times\underline{s} + y\left(s^2\right)\underline{s}$$

$$\underline{\hat{p}} = \left(\underline{z}\cdot\underline{s}t\left(s^2\right) + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)q\left(s^2\right)\right)\underline{x} + \underline{z}n\left(s^2\right) + \underline{z}\times\underline{x}u\left(s^2\right)$$

with some functions $v,\ w,\ y,\ t,\ q,\ n$ and u to be determined and the requirements that $\sigma\left(\underline{\hat{x}}\right) = \underline{x}$ and $\sigma\left(\underline{\hat{p}}\right) = \underline{p}$.

In appendix B.4 we plug the ansätze above for $\underline{\hat{x}}$ and $\underline{\hat{p}}$ into the conditions in (4.16) and (4.17) and then compute the derivatives and commutators. We collect the coefficients of the orthogonal vector basis $\underline{s}$, $\underline{x} \times \underline{s}$, and $\underline{x}$, as well as the coefficients of the one-forms $\underline{\theta}$. In the end of this process we obtain a system of linear ODE's that is easily solvable (and we solve it). The constants of integration of the solution to these ODE's is fixed by the conditions $\sigma\left(\underline{\hat{x}}\right) = \underline{x}$ and $\sigma(\underline{\hat{p}}) = \underline{p}$.

The solutions to these subsequent ODE's associated to the conditions (6.17) and (6.18) for the functions $v,\ w,\ y,\ t,\ q,\ n$ and u give us:

$$\underline{\hat{x}} = \left(\underline{x} - \underline{x} \times \underline{s}\right) \frac{1}{\sqrt{s^2+1}}$$

$$\underline{\hat{p}} = \left(\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{x} + \underline{z}\right) \sqrt{s^2+1}$$

where $\underline{z} = \underline{p} - \underline{x} \times \underline{k}$ and with the following computed conditions holding:

$$\sigma\left(\underline{\hat{x}}\right) = \underline{x} \quad , \quad \sigma(\underline{\hat{p}}) = \underline{p}$$

$$\underline{\hat{x}} \cdot \underline{\hat{x}} = 1/C \quad , \quad \underline{\hat{p}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{p}} - 2i\hbar = 0 \tag{4.20}$$

A solution for our basis is:

$$\sigma^{-1}\left(\underline{x}\right) = \underline{\hat{x}} = \left(\underline{x} - \underline{x} \times \underline{s}\right) \frac{1}{\sqrt{s^2+1}} \tag{4.21}$$

$$\sigma^{-1}(\underline{p}) = \underline{\hat{p}} = \left(\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{x} + \underline{z}\right) \sqrt{s^2+1} \tag{4.22}$$

We now define $\underline{\hat{L}}$ because by doing so, the commutators of $\underline{\hat{L}}$ and $\underline{\hat{x}}$ will contain the angular momentum commutation relations. Define:

$$\underline{\hat{L}} = \frac{1}{2}\left(\underline{\hat{x}} \times \underline{\hat{p}} - \underline{\hat{p}} \times \underline{\hat{x}}\right) = \underline{x} \times \underline{z} + \left(\underline{z} \cdot \underline{s}\right) \underline{x} - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{s} \tag{4.23}$$

$$\underline{\hat{x}} = \left(\underline{x} - \underline{x} \times \underline{s}\right) \frac{1}{\sqrt{s^2+1}} \tag{4.24}$$

along with the computed conditions:

$$\underline{\hat{x}} \cdot \underline{\hat{x}} = 1 \quad , \quad \underline{\hat{L}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{L}} = 0 \quad , \quad \sigma\left(\underline{\hat{L}}\right) = \underline{L} = \underline{x} \times \underline{p} \tag{4.25}$$

Since we have a some basis $\underline{\hat{x}}$ and $\underline{\hat{L}}$, we can define the algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$ (defined in (3.13)) by the image of all $f \in C^{\infty}_{A}\left(T^{*}\mathbb{S}^{2}\right)$ into $\mathcal{Q}_{D,\hat{D}}$ given by σ^{-1} defined by:

$$\begin{aligned} f\left(x, L\right) &= \sum_{(l)(m)} f^{\nu_1\cdots\nu_m}_{(l)(m)\mu_1\cdots\mu_l} x^{\mu_1}\cdots x^{\mu_l} L_{\nu_1}\cdots L_{\nu_m} \\ \overset{\sigma}{\leftrightarrow} \hat{f}\left(\hat{x}, \hat{L}\right) &= \sum_{(l)(m)} f^{\nu_1\cdots\nu_m}_{(l)(m)\mu_1\cdots\mu_l} SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{L}_{\nu_1}\cdots\hat{L}_{\nu_m}\right) \end{aligned}$$

where SYM, defined in (3.22), is the symmetric part of the monomial. As before in section 3.5, $\hat{f}$ is Hermitian iff $\hat{x}^{\mu}$ and $\hat{L}_{\nu}$ so we define that $\hat{x}^{\mu}$,$\hat{L}_{\nu} \in \mathcal{A}_{D,\hat{D}}$ (the Hermitian subalgebra of $\mathcal{Q}_{D,\hat{D}}$).

It is now a simple matter of using the operators in (4.24) and (4.23) to compute the commutators:

$$\begin{aligned} \left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] &= 0 \\ \left[\hat{x}^{\mu}, \hat{L}_{\nu}\right] &= i\hbar\varepsilon^{\mu}{}_{\nu\rho}\hat{x}^{\rho} \\ \left[\hat{L}_{\mu}, \hat{L}_{\nu}\right] &= i\hbar\varepsilon^{\rho}{}_{\mu\nu}\hat{L}_{\rho} \end{aligned} \tag{4.26}$$

Clearly we see that the $\hat{L}$'s generate the standard angular momentum algebra and the $\hat{x}$'s transform properly under rotations. However, both the $\hat{x}$'s and the $\hat{L}$'s form a constrained version of the standard $\mathbb{R}^3$ Euclidean algebra because of the constraints given by (4.25).

5.0 AN ANSATZ FOR THE GLOBAL BUNDLE CONNECTION FOR A GENERAL COTANGENT BUNDLE

In this section, we derive some useful formulas including an ansatz for a solution to $\hat{D}$ subject to the condition in (3.8) for the case when the symplectic manifold is a cotangent bundle. Also, in this section we don't have constraints, which simplifies the computations.

In section 5.1, we derive the cotangent lift of an arbitrary Levi-Civita connection and in section 5.2 we use it as well as the symplectic form to define Weyl Heisenberg bundle. Finally in section 5.3 we derive an ansatz for a solution to the condition in (3.8). We then show that the resulting condition (5.15) is locally integrable by the Cauchy-Kovalevskaya theorem.

5.1 DEFINING THE PHASE-SPACE CONNECTION

As we have mentioned already, the starting place of the Fedosov algorithm (in **step 1**) is the symplectic form ω and a choice of a phase-space connection D. In the present section, we construct a phase-space connection that is the (unique) cotangent lift of the Levi-Civita connection. This choice is made because the resulting phase-space connection will be torsion-free and metric preserving, two very natural conditions. Additionally, this construction is more straightforward than the two-sphere cotangent lift, because we assume here that there are no constraints.

Let ∇ be a Levi-Civita connection on the configuration space M (a general manifold) equipped with a metric g. The connection ∇_a and its curvature operator $\nabla_{[a}\nabla_{b]}$ are defined in some coordinates x^μ as:

$$\nabla_\sigma f(x) = \frac{\partial f}{\partial x^\sigma} \tag{5.1}$$
$$\nabla_\sigma (dx^\mu) = -\Gamma^\mu{}_{\nu\sigma} dx^\nu$$
$$\nabla_\sigma \left(\frac{\partial}{\partial x^\mu} \right) = \Gamma^\nu{}_{\mu\sigma} \frac{\partial}{\partial x^\nu}$$

$$\Gamma^\rho{}_{\mu\nu} = -\frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \tag{5.2}$$
$$\nabla_{[\sigma} \nabla_{\rho]} (dx^\mu) = R^\mu{}_{\nu\sigma\rho} dx^\nu$$

$$R^\mu{}_{\nu\sigma\rho} = -\partial_{[\sigma} \Gamma^\mu{}_{\rho]\nu} + \Gamma^\kappa{}_{\nu[\sigma} \Gamma^\mu{}_{\rho]\kappa} \tag{5.3}$$

where $R^a{}_{bce}$ is the Riemann tensor. Of course we have the conditions that ∇ preserves the metric g and is torsion-free:

$$\nabla_a g_{bc} = 0 \quad , \quad \nabla_{[a} \nabla_{b]} f(x) = 0$$

for all functions $f(x)$. Together these two condition uniquely fix ∇.

In appendix I we compute the cotangent lift of the Levi-Civita connection to a unique phase-space connection. The basic procedure will be explicated now, but the details of the calculation are in appendix I.

The first step in the lifting procedure is to define the phase-space connection D on any set of Darboux coordinates x^μ and p_μ on T^*M to be dx^μ and dp_μ, respectively. Also, we define the action of D on dx^μ to be equal to ∇'s action on dx^μ, i.e.:

$$D \otimes dx^\mu = \nabla \otimes dx^\mu = -\Gamma^\mu{}_{\nu\sigma} dx^\nu \otimes dx^\sigma$$

The hard part is fixing $D \otimes dp_\mu$, but this can be done by using a couple of conditions. The first is that D preserves the symplectic form $\omega = dp_\mu dx^\mu$:

$$D \otimes \omega = 0 \implies (D \otimes dp_\mu) dx^\mu = \left(\Gamma^\sigma{}_{\nu\mu} dx^\nu \otimes dp_\sigma \right) dx^\mu$$

However, this only partially fixes $D \otimes dp_\mu$.

Another condition is needed. We first observe that $p_a = p_\mu dx^\mu$ is a covector for any section in the cotangent bundle ($p_a = w_a(x)$ where $w_a(x)$ is cotangent vector). In other words, we require that $D \otimes dp_\mu$ agrees with:

$$\begin{aligned}\nabla_c \nabla_b p_a &= \nabla_c \left(\partial_b p_a - \Gamma^e{}_{ba} p_e\right) \\ &= \partial_c \left(\partial_b p_a - \Gamma^e{}_{ba} p_e\right) - \Gamma^f{}_{cb} \left(\partial_f p_a - \Gamma^e{}_{fa} p_e\right) - \Gamma^f{}_{ca} \left(\partial_b p_f - \Gamma^e{}_{bf} p_e\right)\end{aligned}$$

First we express the above formula in numerical (greek) indices. Then we replace all terms $(\partial_\nu p_\mu)\, dx^\nu$ by dp_μ. Also, $\partial_c f = \partial f / \partial x^c$ for $f \in C^1(M)$. This condition along with the previous one ($D \otimes \omega = 0$) and the torsion-free condition ($D^2 f = 0$) uniquely fix $D \otimes dp_\mu$.

The cotangent lift of the Levi-Civita connection ∇ is the phase-space connection D given by:

$$Dx^\mu := dx^\mu \tag{5.4}$$

$$Dp_\mu := dp_\mu$$

$$D \otimes dx^\mu = -\Gamma^\mu{}_{\sigma\nu} dx^\nu \otimes dx^\sigma$$

$$D \otimes \alpha_\mu = \Theta^B \otimes D_B \alpha_\mu := -\frac{4}{3} R^\psi{}_{(\mu\sigma)\beta} p_\psi dx^\beta \otimes dx^\sigma + \Gamma^\nu{}_{\mu\sigma} dx^\sigma \otimes \alpha_\nu$$

$$\alpha_\mu := dp_\mu - \Gamma^\nu{}_{\mu\rho} dx^\rho p_\nu \tag{5.5}$$

and the corresponding curvature:

$$D^2 x^\mu = 0 \tag{5.6}$$

$$D^2 p_\mu = 0$$

$$D^2 \otimes dx^\mu = dx^\sigma dx^\rho \otimes R^\mu{}_{\nu\sigma\rho} dx^\nu$$

$$D^2 \otimes \alpha_\mu = \frac{4}{3} dx^\sigma \left(C^\psi{}_{\mu\beta\nu\sigma} p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma} \alpha_\nu\right) \otimes dx^\beta - R^\nu{}_{\mu\sigma\beta} dx^\sigma dx^\beta \otimes \alpha_\nu$$

where $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$.

We can extend to higher order tensors by using the Leibnitz rule and the fact that D and ∇ commute with contractions.

5.2 FIBERING THE WEYL-HEISENBERG BUNDLE OVER PHASE-SPACE

In **step 2** of the algorithm in section 3.3, we have defined the Weyl-Heisenberg bundle E with the matrix basis elements $\hat{y}^A$. Let $\hat{y}^A = (s^\mu, k_\mu)$ where the s's are defined to be the first n $\hat{y}$'s and the k's are defined as the last n $\hat{y}$'s.

The commutators determined by the formula (3.6) and $\omega = dp_\mu dx^\mu$ are:

$$[s^\mu, s^\nu] = 0 = [k_\mu, k_\nu] \quad , \quad [s^\mu, k_\nu] = i\hbar\delta^\mu_\nu$$

The formula for connection D acting on $\hat{y}$ in formula (3.7) is then a simple matter—just plug in (5.4) and (5.6) into the equation (3.7) using the definitions of of s and k:

$$Ds^\mu = -\Gamma^\mu{}_{\sigma\nu} dx^\nu s^\sigma \tag{5.7}$$

$$Dk_\mu := -\frac{4}{3} R^\psi{}_{(\mu\sigma)\beta} dx^\beta s^\sigma p_\psi + \Gamma^\nu{}_{\mu\sigma} dx^\sigma k_\nu$$

$$D^2 s^\mu = dx^\psi dx^\sigma R^\mu{}_{\nu\psi\sigma} s^\nu \tag{5.8}$$

$$D^2 k_\mu = \frac{4}{3} dx^\sigma \left(C^\psi{}_{\mu\beta\nu\sigma} p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma} \alpha_\nu \right) s^\beta - R^\nu{}_{\mu\sigma\beta} dx^\sigma dx^\beta k_\nu$$

where again $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$.

5.3 A GENERAL ANSATZ FOR THE GLOBALLY DEFINED BUNDLE CONNECTION

In this section, we derive a refined condition for $\hat{D}$ equivalent to the condition (3.8) in **step 3** in the algorithm in section 3.3.

As before in section 4.3, we may rewrite the condition (3.8):

$$\left(D - \hat{D}\right)^2 \hat{y}^A = 0$$

in a more convenient form as equation (4.8):

$$\Omega - Dr + \hat{d}r + r^2/\left(i\hbar\right) = 0 \tag{5.9}$$

with:

$$\Omega := -\frac{1}{2}\omega_{AC} R_{CEB}{}^{A} \Theta^C \wedge \Theta^E \hat{y}^B \hat{y}^C \tag{5.10}$$

$$\hat{D} = \left[\hat{Q}, \cdot\right]/\left(i\hbar\right) = \hat{d} + \left[r, \cdot\right]/\left(i\hbar\right) \quad , \quad \hat{d} = \left[\omega_{AB}\hat{y}^A\Theta^B, \cdot\right]/\left(i\hbar\right)$$

However, we want to slightly modify the above condition by substituting $\hat{Q} = \omega_{AB}\hat{y}^A\Theta^B + r$ back in to get:

$$\Omega - D\hat{Q} + \hat{Q}^2/\left(i\hbar\right) = 0 \tag{5.11}$$

where:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = \left[\Omega - D\hat{Q} + \hat{Q}^2/\left(i\hbar\right), \hat{y}^A\right]/\left(i\hbar\right) = 0 \tag{5.12}$$

We keep it in the back of our minds that we could add something that commutes with all $\hat{y}$'s to $\Omega - D\hat{Q} + \hat{Q}^2/\left(i\hbar\right)$ and equation (5.12) would still be satisfied.

As before in section 4.3, any solution to the equation (5.11) for $\hat{Q}$ will be a solution for $\hat{Q}$ in the condition (5.12), which is the same as (3.8). Next we want to find a refined ansatz for $\hat{Q}$ in condition (5.11). See appendix G for some technical notes on the form of solutions to the equation (5.11).

First we write down Ω using the formula (5.10):

$$\begin{aligned} \Omega &= -R^{\nu}{}_{\mu\sigma\beta} dx^\sigma dx^\beta k_\nu s^\mu + \frac{2}{3} D\left(R^{\nu}{}_{(\mu\beta)\sigma} p_\nu s^\beta s^\mu dx^\sigma\right) \\ &= -R^{\nu}{}_{\mu\sigma\beta} dx^\sigma dx^\beta k_\nu s^\mu + \frac{2}{3} dx^\sigma \left(C^{\psi}{}_{\mu\beta\nu\sigma} p_\psi dx^\nu + R^{\nu}{}_{(\mu\beta)\sigma}\alpha_\nu\right) s^\beta s^\mu \end{aligned} \tag{5.13}$$

where $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$.

Just as in section 4.3, we verify that Ω gives the curvature as commutators:

$$\frac{1}{i\hbar}\left[\Omega, s^\mu\right] = D^2 s^\mu = R^\mu{}_{\nu\psi\varepsilon} dx^\psi dx^\varepsilon s^\nu$$

$$\frac{1}{i\hbar}\left[\Omega, k_\mu\right] = D^2 k_\mu = \frac{4}{3} dx^\sigma \left(C^\psi{}_{\mu\beta\nu\sigma} p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma}\alpha_\nu \right) s^\beta - R^\nu{}_{\mu\sigma\beta} dx^\sigma dx^\beta k_\nu$$

In appendix D.1 we have put the proof that equation (5.11) imply the ansatz (5.14) and its condition(5.15). The proof begins with an ansatz in (D.2) for $\hat{Q}$:

$$\hat{Q} = \left(k_\nu f^\nu{}_\mu(x,s) + p_\nu g^\nu{}_\mu(x,s) + h_\mu(x,s)\right) dx^\mu + j^\mu(x,s)\,\alpha_\mu$$

Through a series of redefinitions, which are based on symmetries of equation (5.11) for cotangent bundles as well as the ensuing computations, we refined this ansatz to a new ansatz in (5.14) and its condition (5.15). Let $\hat{\partial}_\mu := \partial/\partial s^\mu$, the refined ansatz is:

$$\begin{aligned} \hat{Q} \;=\; & \left(s^\mu \alpha_\mu - z_\mu dx^\mu\right) + j^\mu \alpha_\mu + z_\nu f^\nu{}_\mu dx^\mu \\ & + p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - dx^\sigma \hat{\partial}_\sigma \right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma \right) \end{aligned} \tag{5.14}$$

where $\hat{\partial}_\mu := \partial/\partial s^\mu$ and along with condition on $f^\nu{}_\mu$:

$$\left(\left(D + f^\mu{}_\rho dx^\rho \hat{\partial}_\mu - dx^\mu \hat{\partial}_\mu \right) f^\nu{}_\sigma + \Gamma^\nu{}_{\rho\mu} dx^\mu f^\rho{}_\sigma - \Gamma^\nu{}_{\sigma\mu} dx^\mu + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta \right) dx^\sigma = 0 \tag{5.15}$$

Therefore, the ansatz (5.14) and its condition(5.15) is equivalent to the condition (5.11).

The term in $\hat{Q}$:

$$p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - \hat{\partial}_c \right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho \right)$$

could be a tensorial iff j^μ and f^μ_ν are matrix-valued tensors. This can be seen by writing the above in abstract configuration space indices:

$$p_b \left(\nabla_c j^b + f^e{}_c \hat{\partial}_e - \hat{\partial}_c \right) j^b$$

where we have assumed here that j^b only depends on x and s so that $Dj^b = \Theta^C D_C j^b = \nabla_c j^b$.

Moreover, the equation (5.15) is locally integrable for $f^\nu{}_\mu$ by the Cauchy-Kovalevskaya theorem (see appendix D.2). This fact allows us to come up with an iterative solution in the spirit of the series of Fedosov star-product.

The proof for the local integrability of $f^{\nu}{}_{\sigma} \in E$ begins with assuming $f^{\nu}{}_{\sigma}$ only depends on x and s (and not p or k). This allows us to rewrite the condition (5.15) in abstract indices as:

$$R_{ca}{}^{b}{}_{m}s^{m} + \left(\nabla_{c} + f^{e}{}_{c}\hat{\partial}_{e}\right) f^{b}{}_{a} = 0$$

In appendix D.2, we showed that:

$$\left(\nabla_{[n} + f^{d}{}_{[n}\hat{\partial}_{d}\right)\left(R_{ca]}{}^{b}{}_{m}s^{m} + \left(\nabla_{c} + f^{e}{}_{c}\hat{\partial}_{|e|}\right) f^{b}{}_{a]}\right) = 0$$

The equation above is of the form:

$$P\left(\begin{pmatrix}\text{known}\\ \text{quantity}\end{pmatrix} + Pf\right) = 0$$

where P is a differential operator dependent on $f^{\nu}{}_{\sigma}$ and the original equation is:

$$\begin{pmatrix}\text{known}\\ \text{quantity}\end{pmatrix} + Pf = 0$$

By the Cauchy-Kovalevskaya theorem the equation for $f^{\nu}{}_{\sigma}$ locally integrable.

Now we state the main and final result of this chapter in a theorem:

Thm. Given any cotangent bundle $T^{*}M$, the solution to the equation in (5.11):

$$\Omega - D\hat{Q} + \hat{Q}^{2}/\left(i\hbar\right) = 0 \tag{5.16}$$

is equivalent to (5.14):

$$\begin{aligned}\hat{Q} \;=\;& \left(s^{\mu}\alpha_{\mu} - z_{\mu}dx^{\mu}\right) + j^{\mu}\alpha_{\mu} + z_{\nu}f^{\nu}{}_{\mu}dx^{\mu} \\ &+p_{\nu}\left(\left(D + f^{\sigma}{}_{\rho}dx^{\rho}\hat{\partial}_{\sigma} - dx^{\sigma}\hat{\partial}_{\sigma}\right) j^{\nu} + \Gamma^{\nu}{}_{\rho\sigma}dx^{\sigma}j^{\rho} - \frac{2}{3}R^{\nu}{}_{(\mu\beta)\sigma}s^{\beta}s^{\mu}dx^{\sigma}\right)\end{aligned} \tag{5.17}$$

and the condition on $f^{\nu}{}_{\mu}$ in (5.15):

$$\left(\left(D + f^{\mu}{}_{\rho}dx^{\rho}\hat{\partial}_{\mu} - dx^{\mu}\hat{\partial}_{\mu}\right) f^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\rho\mu}dx^{\mu}f^{\rho}{}_{\sigma} - \Gamma^{\nu}{}_{\sigma\mu}dx^{\mu} + R^{\nu}{}_{\mu\beta\sigma}s^{\mu}dx^{\beta}\right)dx^{\sigma} = 0 \tag{5.18}$$

where the equation (5.18) is locally integrable for $f^{\nu}{}_{\mu}$ by the Cauchy-Kovalevskaya theorem. (Notation $\hat{\partial}_{\mu} := \partial/\partial s^{\mu}$)

6.0 FEDOSOV QUANTIZATION ON CONSTANT CURVATURE MANIFOLDS OF CODIMENSION ONE

In this chapter we present original results of this thesis, calculating the quantization map σ^{-1} for $M_{C_{p,q}}$ as well as analyzing the image $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C_A^{\infty}\left(T^*M_{C_{p,q}}\right)\right)$, i.e., the algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$. This contains the main results of the thesis.

In section 6.1, we review in detail the geometry of these spaces using the analogy with dS/AdS space-times to guide us. Next in section 6.2 we define our phase-space connection and in section 6.4 we construct the globally defined bundle connection $\hat{D}$ by means of an ansatz. This is an exact construction of $\hat{D}$. We construct the Fedosov quantization map σ^{-1} in section 6.5 as prescribed by the algorithm in section 3.3.

The section 6.6 is devoted to constructing the map σ^{-1} for the same manifold with a different embedding. This has no effect on the algebra of observables $\widetilde{\mathcal{A}}_{D,\hat{D}}$, which will be computed as $\mathbb{SO}\left(p+1,q+1\right)$, except that the Casimir invariant of the algebra of observables $\widetilde{\mathcal{A}}_{D,\hat{D}}$ (in equation (6.31) in section 6.8) is dependent on the change. In basic terms it gives flexibility in choosing different representations of $\mathbb{SO}\left(p+1,q+1\right)$.

In section 6.7 we compute the commutators of $\sigma^{-1}\left(x^{\mu}\right)$ and $\sigma^{-1}\left(p_{\mu}\right)$ as before. Next in section 6.8, we reorganize the generators of $\widetilde{\mathcal{A}}_{D,\hat{D}}$, i.e., $\sigma^{-1}\left(x^{\mu}\right)$ and $\sigma^{-1}\left(p_{\mu}\right)$, and find that $\widetilde{\mathcal{A}}_{D,\hat{D}} = \mathbb{SO}\left(p+1,q+1\right)$. Using the program in 3.4, in sections 6.9 (the dS/AdS case) and 6.10 (a more general case) we find the differential equation on $\phi\left(x\right) = \langle x|\phi\rangle$ coming from $\sigma^{-1}\left(H\right)|\phi\rangle = 0$ where $H = p^2 - m^2$ is the Hamiltonian associated to geodesic motion of a single free particle on $M_{C_{p,q}}$.

6.1 THE BACKGROUND GEOMETRY

Before we go into the details of the results we first want to review the geometry of constant curvature manifolds of codimension one. To this end, we rely on the fact that the geometry in the more general case is a relatively straightforward generalization of the familiar two-sphere and dS/AdS manifolds. The main motivation for considering this class of manifolds was the fact that the sphere and dS/AdS lie in it.

We start with the phase space of a single classical particle confined to a constant curvature manifold with metric $\left(M_{C_{p,q}}, g\right)$ that is imbedded in $(\mathbb{R}^{n+1}, \eta)$ where $\dim M_{C_{p,q}} = p+q = n$ and η is a pseudoeuclidean metric. The imbedding specifically is the hyperboloid:

$$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = 1/C$$

η induces a metric on $M_{C_{p,q}}$ called g and explicitly:

$$g_{\mu\nu} := \eta_{\mu\nu} - C x_\mu x_\nu \tag{6.1}$$

which is easily obtained by the constraint above (just project each index orthogonal to x, $T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \to T^{\mu\mu_2\cdots\mu_n}{}_{\nu_1\cdots\nu_m}\left(\delta^{\mu_1}_{\mu} - C x_\mu x^{\mu_1}\right)$ etc.). We will always raise and lower the lower-case indices or $M_{C_{p,q}}$ indices (greek or Latin) by the metric of the imbedding space $\mathbb{R}^{n+1}$ η.

We make the convention that the positive signature directions are the "time" directions while the negative ones are the "space" directions. The signature of g denoted by $sign\,(g)$ is (p,q), then for $C>0$ (this space-time is denoted by $M^+_{C_{p,q}}$), η is a pseudoeuclidean metric of signature $(p+1,q)$ or explicitly:

$$\eta = diag(\underbrace{1,\ldots,1}_{p+1},\underbrace{-1,\ldots,-1}_{q})$$

This is in contrast to the $C<0$ case where η is a pseudo-euclidean metric of signature $(p,q+1)$. This is because for $C>0$ (this space-time is denoted by $M^-_{C_{p,q}}$) the hyperboloid is "time"-like, i.e., it has normal vectors pointing in a combination of the $p+1$ positive signature directions. Thus, the induced metric has a signature of one less "time" dimension from the embedding. For the case of $C<0$ the hyperboloid is space-like and thus the induced metric has a signature of one less "space" dimension, i.e., it has normal vectors pointing in a combination of the $q+1$ negative signature directions.

A good way to visualize these spaces is to look at the $1+3$ dimensions, which gives us the familiar de Sitter (dS) and Anti-de Sitter (AdS) space-times for $C<0$ and $C>0$ respectively. The picture, of course, generalizes very naturally. The embeddings in these cases are:

$$\left(x^0\right)^2 - \left(x^4\right)^2 - \underline{x}\cdot\underline{x} = 1/C \quad , \quad C<0 \tag{6.2}$$

$$\left(x^0\right)^2 + \left(x^4\right)^2 - \underline{x}\cdot\underline{x} = 1/C \quad , \quad C>0 \tag{6.3}$$

where:

$$\underline{x} = \left(x^1, x^2, x^3\right)$$

Figures 6.1 and 6.2 show the pictures of dS and AdS respectively.

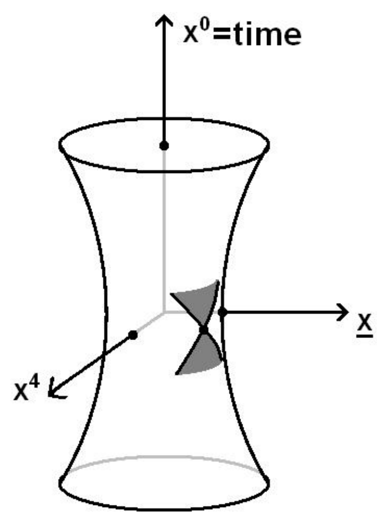

Figure 6.1: dS space-time ($C<0$). An arbitrary light cone is drawn with the shaded region being the timelike direction.

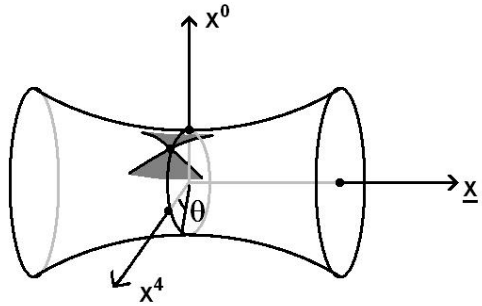

Figure 6.2: AdS space-time ($C > 0$). An arbitrary light cone is drawn with the shaded region being the timelike direction.

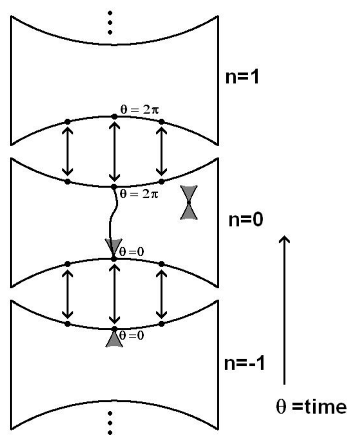

Figure 6.3: The covering space of dS space-time. Here the observer who follows the world-line starting at $\theta = 0$ and ending at $\theta = 2\pi$ does not arrive back in his past. An arbitrary light cone is drawn with the shaded region being the timelike direction.

We notice that in the case of dS the definition of time must be x^0 and in AdS it must be the 0-4 angle θ. We immediately notice a problem in this embedding of AdS: If we follow a world line starting at $\theta = 0$ and ending at $\theta = 2\pi$ we arrive back at our starting point. We assume that we cannot reach the past by going far into the future. This is to avoid serious paradoxes of what must be a pathological space-time.

The resolution to this dilemma is to go to the covering space of the hyperboloid by "unidentifying" (or not identifying them in the first place) the values 0, $\pm 2\pi$, $\pm 4\pi$,.... This is done by breaking the hyperboloid into leaves (labelled by n). So if we follow a world-line starting at $\theta = 0$, when we get to 2π we will be in a different leaf of the covering space and thus not at our original point. The picture is described by first imagining that we have infinitely many hyperboloids. We then cut them length-wise, open them up, and put each successive one above the other, as shown in Figure 6.3. Thus the topology of time is $\mathbb{R}$ not an $\mathbb{S}^1$.

By differentiating $x^\mu x_\mu = 1/C$ we may obtain the condition on p_μ:

$$2dx^\mu x_\mu = 0 \implies x^\mu p_\mu = 0$$

The embedding formulas are then:

$$x^\mu x_\mu = 1/C \quad , \quad x^\mu p_\mu = 0 \tag{6.4}$$

where C is an arbitrary real constant.

6.2 DEFINING THE PHASE-SPACE CONNECTION

In this section we will state our choice of our phase-space connection D used in the Fedosov algorithm for a constant curvature manifold embeddable in a flat space of codimension one. We use the cotangent lift of the Levi-Civita connection associated to the metric g in (6.1) given in (5.4). The quantities we need are $g_{\mu\nu}$, $\Gamma^\rho{}_{\mu\nu}$, and $R^\mu{}_{\nu\sigma\rho}$. However, because we have constraints, deriving the Levi-Civita connection and the associated curvature isn't as straightforward as just plugging g into the standard formula in (5.2):

$$\Gamma^\rho{}_{\mu\nu} = -\frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

Just as the case for the two-sphere in section 4.1, we have redundancy because we are expressing an n-dimensional manifold in an $(n+1)$-dimensional coordinate system. To fix this redundancy and to get an intrinsic phase-space connection D, we impose constraints that D preserves the constraint equations like the two-sphere:

$$\underbrace{\nabla\otimes\cdots\otimes\nabla}_{l}\left(\eta_{\mu\nu}x^{\mu}x^{\nu}-1/C\right)=0 \tag{6.5}$$

for all $l\in\mathbb{Z}^{+}$.

$$x^{\mu}x_{\mu}=1/C \quad , \quad x^{\mu}p_{\mu}=0 \tag{6.6}$$

and the subsequent conditions derived from $D\left(\eta_{\mu\nu}x^{\mu}x^{\nu}-1/C\right)=0$ and $D\left(x^{\mu}p_{\mu}\right)=0$:

$$x^{\mu}dx_{\mu}=0 \quad , \quad p_{\mu}dx^{\mu}+x^{\mu}dp_{\mu}=0 \tag{6.7}$$

The constraint $x^{\mu}dx_{\mu}=0$ means that a formula such as:

$$\nabla\otimes dx^{\mu}=-\Gamma^{\mu}{}_{\sigma\nu}dx^{\nu}\otimes dx^{\sigma}$$

is ambiguous because it is invariant under the change:

$$\Gamma^{\rho}{}_{\mu\nu}\rightarrow\tilde{\Gamma}^{\rho}{}_{\mu\nu}=\Gamma^{\rho}{}_{\mu\nu}+x^{\rho}q_{\mu\nu}+x_{(\mu}f^{\rho}{}_{\nu)}$$

where $q_{\mu\nu}=q_{(\mu\nu)}$ and $f^{\rho}_{\ \nu}$ are arbitrary (the symmetrization of $x_{(\mu}f^{\rho}{}_{\nu)}$ is to preserve the torsion-free condition).

In appendix E.1.1 we use the constraints in (6.6) and (6.7) to fix $q_{\mu\nu}$ and $f^{\rho}{}_{\nu}$. However, we are left with some additional freedom which will not affect any of our formulas, so we make an arbitrary choice here (this refers to the freedom in choosing B and F in (E.16) and (E.17) in appendix E.1.1). The result will of the calculation is:

$$\Gamma^{\mu}{}_{\nu\sigma}=Cx^{\mu}g_{\nu\sigma}-2Cx_{(\nu}\left(\delta^{\mu}_{\sigma)}-Cx_{\sigma)}x^{\mu}\right) \tag{6.8}$$

In appendix E.1.2 we compute the curvature in a similar way and here is a summary of it. Like $\Gamma^{\mu}{}_{\nu\sigma}$, the curvature tensor will have a redundancy:

$$R^{\mu}{}_{\nu\sigma\rho}\rightarrow\tilde{R}^{\mu}{}_{\nu\sigma\rho}=R^{\mu}{}_{\nu\sigma\rho}+x_{\nu}q^{\mu}{}_{\sigma\rho}+x_{\sigma}f^{\mu}{}_{\nu\rho}+x_{\rho}h^{\mu}{}_{\nu\sigma}$$

However, since this is a tensor expressible in the basis (dx^μ, α_μ) each index is projected orthogonal to x, i.e. for some tensor $T^{\mu_1 \cdots \mu_l}{}_{\nu_1 \cdots \nu_m}$ the projection is $l+m$ conditions:

$$\eta_{\mu\nu} T^{\mu\mu_2 \cdots \mu_l}{}_{\nu_1 \cdots \nu_m} x^\nu = 0 \quad , \ldots , \quad \eta_{\mu\nu} T^{\mu_1 \cdots \mu_{l-1}\mu}{}_{\nu_1 \cdots \nu_m} x^\nu = 0$$

$$T^{\mu\mu_2 \cdots \mu_l}{}_{\nu\nu_2 \cdots \nu_m} x^\nu = 0 \quad , \ldots , \quad T^{\mu_1 \cdots \mu_{l-1}\mu}{}_{\nu_1 \cdots \nu_{m-1}\nu} x^\nu = 0$$

The result for the curvature is:

$$R^\mu{}_{\nu\sigma\rho} = -C \left(\delta^\mu_{[\sigma} - C x_{[\sigma} x^\mu \right) g_{\rho]\nu}$$

Also, to obtain the components of the symplectic form ω in the basis, (dx^μ, α_μ) we project each index orthogonal to x.

$$\omega = \left(\delta^\mu_\nu - C x^\mu x_\nu \right) \alpha_\mu dx^\nu$$

By a simple substitution of the formula for (6.8) into the constraint in (6.7), this constraint can be rewritten as:

$$x^\mu dx_\mu = 0 \quad , \quad \alpha_\mu dx^\mu = 0 \tag{6.9}$$

where α_μ is defined in (5.5) as $\alpha_\mu := dp_\mu - \Gamma^\nu{}_{\mu\rho} dx^\rho p_\nu$.

Therefore, we have all quantities needed for the cotangent lift of the Levi-Civita connection ∇ (associated to the metric g) by putting the quantities:

$$g_{\mu\nu} = \eta_{\mu\nu} - C x_\mu x_\nu \tag{6.10}$$

$$\Gamma^\mu{}_{\nu\sigma} = C x^\mu g_{\nu\sigma} - 2C x_{(\nu} \left(\delta^\mu_{\sigma)} - C x_{\sigma)} x^\mu \right)$$

$$R^\mu{}_{\nu\sigma\rho} = -C \left(\delta^\mu_{[\sigma} - C x_{[\sigma} x^\mu \right) g_{\rho]\nu}$$

$$\omega = \left(\delta^\mu_\nu - C x^\mu x_\nu \right) \alpha_\mu dx^\nu$$

$$\alpha_\mu := dp_\mu - \Gamma^\nu{}_{\mu\rho} dx^\rho p_\nu$$

into the formula (5.4).

We also have the constraints:

$$x^\mu x_\mu = 1/C \quad , \quad x^\mu p_\mu = 0 \tag{6.11}$$

$$x^\mu dx_\mu = 0 \quad , \quad \alpha_\mu dx^\mu = 0 \tag{6.12}$$

On a general technical note, we will proceed in an identical fashion for most of the paper: at each step we will verify that all relevant constraints are satisfied. Although we choose a set of coordinates x^μ, even ones with constraints, objects such as ∇, g, etc. are intrinsic and coordinate-free objects.

6.3 FIBERING THE WEYL-HEISENBERG BUNDLE OVER PHASE-SPACE

In this section we implement **step 2** of the algorithm in section 3.3. The relations defining the $\hat{y}$'s (relations (3.6) and (3.7) in **step 2**) are:

$$\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB}$$

$$D\hat{y}^A = -\Gamma^A{}_B\hat{y}^B = -\Gamma^A{}_{BC}\Theta^C\hat{y}^B$$

Our choice for $\Theta^A = (dx^\mu, \alpha_\mu)$ where α_μ is defined in (6.10) and again, let $\hat{y}^A = (s^\mu, k_\mu)$ where the s's are defined to be the first $n+1$ $\hat{y}$'s and the k's are defined as the last $n+1$ $\hat{y}$'s.

From the definition of $\hat{y}$ the commutation relations in (3.6) and from the formula (6.10):

$$[s^\mu, s^\nu] = 0 = [k_\mu, k_\nu] \ \ , \ \ [s^\mu, k_\nu] = i\hbar\left(\delta^\mu_\nu - Cx^\mu x_\nu\right)$$

Since dx^μ and α_μ are perpendicular to x, the matrix counterparts s^μ and k_μ are as well:

$$\eta_{\mu\nu}x^\mu s^\nu = x^\mu k_\mu = 0 \tag{6.13}$$

To get Ds^μ and Dk_μ, we plug the formulas (6.10) into (5.7) and (5.8):

$$Ds^\mu = -\Gamma^\mu{}_{\sigma\nu}dx^\nu s^\sigma \tag{6.14}$$

$$Dk_\mu := -\frac{4}{3}R^\psi{}_{(\mu\sigma)\beta}dx^\beta s^\sigma p_\psi + \Gamma^\nu{}_{\mu\sigma}dx^\sigma k_\nu$$

$$D^2s^\mu = dx^\psi dx^\sigma R^\mu{}_{\nu\psi\sigma}s^\nu \tag{6.15}$$

$$D^2k_\mu = \frac{4}{3}dx^\sigma\left(C^\psi{}_{\mu\beta\nu\sigma}p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma}\alpha_\nu\right)s^\beta - R^\nu{}_{\mu\sigma\beta}dx^\sigma dx^\beta k_\nu$$

where again $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$.

Note: These s^μ and k_μ are different than the two-sphere $\underline{s}$ and $\underline{k}$. The relation between them is:

$$\underline{s}_{\mathbb{S}^2} = \underline{x} \times \underline{s}_{C_{3,0}} \quad , \quad \underline{k}_{\mathbb{S}^2} = \underline{x} \times \underline{k}_{C_{3,0}}$$

This is because:

$$\underline{\theta}_{\mathbb{S}^2} = \underline{x} \times d\underline{x} \quad , \quad \underline{\alpha}_{\mathbb{S}^2} = \underline{x} \times \underline{\alpha}_{C_{3,0}}$$

6.4 CONSTRUCTING THE GLOBALLY DEFINED BUNDLE CONNECTION

In **step 3** in the algorithm in section section 3.3, we must determine the global derivation $\hat{D} = \left[\hat{Q}, \cdot\right]$, i.e., solve for $\hat{Q} \in E \otimes \Lambda$. To this end, we use the previously derived ansatz for $\hat{Q}$ in (5.17) for the case of a general cotangent bundle. All we have to do next is find a solution to the condition on it (5.18) and the way we solve for $\hat{Q}$ is by generalizing the solution to the two-sphere case [1] in (4.14). In appendix C.1.1, we show that (4.14) is related to the ansatz in (5.17) when $f^{\nu}{}_{\mu} = Cs^{\nu}s_{\mu}$ and $j^{\mu} = 0$, namely that:

$$\hat{Q}_{C_{3,0}} = \hat{Q}_{\mathbb{S}^2} + \underline{p} \cdot (\underline{x} \times \underline{\theta})$$

Since $\hat{Q}$ is a graded commutator, the term $\underline{p} \cdot (\underline{x} \times \underline{\theta})$ commutes (in a graded way) with everything. Therefore the two $\hat{D}$'s, $\hat{D}_{C_{3,0}} = \left[\hat{Q}_{C_{3,0}}, \cdot\right]$ and $\hat{D}_{\mathbb{S}^2} = \left[\hat{Q}_{\mathbb{S}^2}, \cdot\right]$ are equal.

We quickly realize that $f^{\nu}{}_{\mu} = Cs^{\nu}s_{\mu}$ is a solution to (5.18) in case of *all* constant curvature manifolds embeddable in a flat space of codimension one. In appendix C.1 we explicitly show that $f^{\nu}{}_{\mu} = Cs^{\nu}s_{\mu}$ is a solution for this more general case. Therefore, the solution for $\hat{Q}$ in the case of all constant curvature manifolds embeddable in a flat space of codimension one is:

$$\hat{Q} = (s^{\mu}\alpha_{\mu} - z_{\mu}dx^{\mu}) - C\,(z_{\nu}s^{\nu})\,(s_{\mu}dx^{\mu}) + \frac{C}{3}\left((p_{\nu}s^{\nu})\,(s_{\mu}dx^{\mu}) - (p_{\nu}dx^{\nu})\,u\right) \tag{6.16}$$

[1] Be very careful here, the s's, k's and α's in the sphere case are related to this case by $\underline{s}_{\mathbb{S}^2} = \underline{x} \times \underline{s}_{C_{3,0}}$, $\underline{k}_{\mathbb{S}^2} = \underline{x} \times \underline{k}_{C_{3,0}}$, $\underline{\theta}_{\mathbb{S}^2} = \underline{x} \times d\underline{x}$, and $\underline{\alpha}_{\mathbb{S}^2} = \underline{x} \times \underline{\alpha}_{C_{3,0}}$.

6.5 DEFINING THE SECTION IN THE BUNDLE: OUR OBSERVABLE ALGEBRA

In this section we will construct the map σ^{-1} as in **step 4** in section 3.3. Just as in the case for $T^*\mathbb{R}^n$ in section 3.5 and the two-sphere in section 4.4 we construct $\widetilde{\mathcal{A}}_{D,\hat{D}} := \sigma^{-1}\left(C_A^\infty\left(T^*M_{C_{p,q}}\right)\right)$ (see the definition in (3.13)) where $C_A^\infty\left(T^*M_{C_{p,q}}\right)$ is the space of analytic functions in this section. Again, see in section 3.7 how σ^{-1} is extended to all $C^\infty\left(T^*M_{C_{p,q}}\right)$.

The algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$ is simply the enveloping algebra of some basis $\sigma^{-1}\left(x^\mu\right)$ and $\sigma^{-1}\left(p_\mu\right)$ and the goal now is to find a suitable definition of these elements. Explicitly we want some $\hat{x}^\mu, \hat{p}_\nu \in E$ such that the following conditions hold:

$$\left(D-\hat{D}\right)\hat{x}^\mu = 0 \quad , \quad \sigma\left(\hat{x}^\mu\right) := x^\mu \tag{6.17}$$

$$\left(D-\hat{D}\right)\hat{p}_\mu = 0 \quad , \quad \sigma\left(\hat{p}_\mu\right) := p_\mu \tag{6.18}$$

To find an exact solution, we make the ansatz for the solutions:

$$\hat{x}^\mu = f\left(u\right)x^\mu + h\left(u\right)s^\mu$$

$$\hat{p}_\mu = z_\nu s^\nu x_\mu g\left(u\right) + z_\mu j\left(u\right)$$

where $u := \eta_{\mu\nu}s^\mu s^\nu$ and $z_\mu := k_\mu + p_\mu$.

In appendix C.2.1 we put these ansätze into the conditions (6.17) and (6.18). By taking the derivatives, we obtain differential equations for f, h, g, and j. The constants of integration are subsequently fixed by the conditions $\sigma\left(\hat{x}^\mu\right) := x^\mu$ and $\sigma\left(\hat{p}_\mu\right) := p_\mu$. The solutions we obtain:

$$\hat{x}^\mu = \left(x^\mu + s^\mu\right)\frac{1}{\sqrt{Cu+1}} \tag{6.19}$$

$$\hat{p}_\mu = \left(-Cz_\nu s^\nu x_\mu + z_\mu\right)\sqrt{Cu+1} - iC\hbar n\hat{x}_\mu \tag{6.20}$$

where $u = s_\mu s^\mu$, $z_\mu := k_\mu + p_\mu$, and with the computed conditions:

$$\sigma\left(\hat{x}^\mu\right) = x^\mu \quad , \quad \sigma\left(p_\mu\right) = \hat{p}_\mu$$

$$\hat{x}\cdot\hat{x}=1/C \quad , \quad \hat{x}\cdot\hat{p}=\hat{p}\cdot\hat{x}-ni\hbar=0 \tag{6.21}$$

We define these as our basis:

$$\sigma^{-1}(x^{\mu})=\hat{x}^{\mu} \quad , \quad \sigma^{-1}(\hat{p}_{\mu})=p_{\mu}$$

6.6 CHANGE OF EMBEDDING

In this section we want to find $\hat{x}$ and $\widehat{\tilde{p}}$ associated to the embedding:

$$x^{\mu}x_{\mu}=1/C \quad , \quad x^{\mu}\tilde{p}_{\mu}=A$$

knowing the ones for $A=0$ as was constructed in the last section.

The main motivation for doing so is that in sections section 6.8 the Casimir invariant computed in equation (6.31) is:

$$\hat{M}_{\mu'\nu'}\hat{M}^{\mu'\nu'}=-\frac{1}{2}(A-i\hbar n)A$$

where $\hat{M}_{\mu'\nu'}$ will be the basis of our analytic observable algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$. Allowing for A to be nonzero allows for more flexibility to choose different representations of the algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$ which is the group $\mathbb{SO}(p+1,q+1)$, i.e., $\widetilde{\mathcal{A}}_{D,\hat{D}}=\mathbb{SO}(p+1,q+1)$.

To find them, we exploit the canonical transformation:

$$\tilde{p}_{\mu}=p_{\mu}+CAx_{\mu} \quad , \quad \tilde{x}^{\mu}=x^{\mu}$$

because it leaves the symplectic form invariant:

$$\tilde{\omega}=d\tilde{p}_{\mu}d\tilde{x}^{\mu}=dp_{\mu}dx^{\mu}=\omega$$

therefore, it also leaves D and $\hat{D}$ unchanged. Subsequently, the two solutions:

$$\hat{x}^{\mu}=(x^{\mu}+s^{\mu})\frac{1}{\sqrt{Cu+1}}$$

$$\hat{p}_{\mu}=(-Cz_{\nu}s^{\nu}x_{\mu}+z_{\mu})\sqrt{Cu+1}-iC\hbar n\hat{x}_{\mu}$$

are still solutions to the equations (6.19) and (6.20):

$$\left(D-\hat{D}\right)\hat{x}^{\mu}=0 \quad , \quad \sigma\left(\hat{x}^{\mu}\right):=x^{\mu}$$

$$\left(D-\hat{D}\right)\hat{p}_{\mu}=0 \quad , \quad \sigma\left(\hat{p}_{\mu}\right):=p_{\mu}$$

Now we want $\widehat{\tilde{p}}_{\mu}$, so we perform the canonical transformation:

$$x^{\mu}=x^{\mu} \quad , \quad p_{\mu}=\tilde{p}_{\mu}-CAx_{\mu}$$

Thus:

$$\hat{x}^{\mu}=\sigma^{-1}\left(x^{\mu}\right)=\left(x^{\mu}+s^{\mu}\right)\frac{1}{\sqrt{Cu+1}} \tag{6.22}$$

$$\widehat{\tilde{p}}_{\mu}:=\sigma^{-1}\left(\tilde{p}_{\mu}\right)=\hat{p}_{\mu}+CA\hat{x}_{\mu}=\left(z_{\mu}-Cz_{\nu}s^{\nu}x_{\mu}\right)\sqrt{Cu+1}-C\left(i\hbar n-A\right)\hat{x}_{\mu} \tag{6.23}$$

with computed conditions:

$$\eta_{\mu\nu}\hat{x}^{\mu}\hat{x}^{\nu}=1/C \quad , \quad \hat{x}^{\mu}\widehat{\tilde{p}}_{\mu}=\widehat{\tilde{p}}_{\mu}\hat{x}^{\mu}-ni\hbar=A \tag{6.24}$$

In group theoretic terminology the two conditions above represent the Casimir invariants of the quantum algebra $\mathcal{Q}_{D,\hat{D}}$.

Important: From now on we will use $\widehat{\tilde{p}}_{\mu}$ and drop the tilde, i.e. we will write it as $\hat{p}_{\mu}$.

6.7 THE COMMUTATORS

Now it is a very straightforward matter of working out the commutators $[\hat{x}^\mu, \hat{x}^\nu]$, .$[\hat{x}^\mu, \hat{p}_\nu]$, and $[\hat{p}_\mu, \hat{p}_\nu]$ using the formulas in (6.19) and (6.20):

$$[\hat{x}^\mu, \hat{x}^\nu] = 0 \tag{6.25}$$

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar \left(\delta^\mu_\nu - C\hat{x}^\mu \hat{x}_\nu\right)$$

$$[\hat{p}_\mu, \hat{p}_\nu] = 2i\hbar C \hat{x}_{[\nu}\hat{p}_{\mu]}$$

along with the computed conditions (see appendix C.3 for details):

$$\hat{x}^\mu \hat{x}_\mu = 1/C \quad , \quad \hat{p}_\mu \hat{x}^\mu + ni\hbar = \hat{x}^\mu \hat{p}_\mu = A$$

We now define:

$$\hat{M}_{\mu\nu} = \hat{x}_{[\mu}\hat{p}_{\nu]} = \hat{p}_{[\nu}\hat{x}_{\mu]} = \left(-Cz_\rho s^\rho x_{[\nu} + z_{[\nu}\right)\left(x_{\mu]} + s_{\mu]}\right)$$

The projection of $\hat{M}$ is found to be:

$$\sigma\left(\hat{M}_{\mu\nu}\right) = x_{[\mu}p_{\nu]} = M_{\mu\nu}$$

We recognize that $\hat{M}$ and $\hat{x}$ are the more "natural" variables than $\hat{x}$ and $\hat{p}$ because $\hat{p}_\mu \hat{x}^\mu = -ni\hbar$ and $\hat{x}^\mu \hat{p}_\mu = A$ where A is an arbitrary constant. These are very "unnatural" since there is no reason why it shouldn't be $\hat{p}_\mu \hat{x}^\mu = A$ and $\hat{x}^\mu \hat{p}_\mu = ni\hbar$ or something else like this. $\hat{M}$ projects out the part of the momentum $\hat{p}$ that is parallel to $\hat{x}$ ($2\hat{x}^\mu \hat{M}_{\mu\nu} = \hat{p}_\nu/C - A\hat{x}_\nu$). We regard this part of $\hat{p}$ to be irrelevant because it does not affect the form of the commutators in (6.25) and it preserves the symplectic form.

We have the definitions:

$$\hat{x}^\mu = (x^\mu + s^\mu)\frac{1}{\sqrt{Cu+1}} \tag{6.26}$$

$$\hat{M}_{\mu\nu} = \hat{x}_{[\mu}\hat{p}_{\nu]} = \hat{p}_{[\nu}\hat{x}_{\mu]} = -Cz_\rho s^\rho x_{[\nu}s_{\mu]} + z_{[\nu}x_{\mu]} + z_{[\nu}s_{\mu]}$$

and the computed commutation relations (see appendix C.4 for details):

$$[\hat{x}^\mu, \hat{x}^\nu] = 0 \tag{6.27}$$

$$\left[\hat{x}_\mu, \hat{M}_{\nu\rho}\right] = i\hbar\hat{x}_{[\nu}\eta_{\rho]\mu}$$

$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = i\hbar\left(\hat{M}_{\sigma[\mu}\eta_{\nu]\rho} - \hat{M}_{\rho[\mu}\eta_{\nu]\sigma}\right)$$

subject to the conditions:

$$\hat{x}^\mu\hat{x}_\mu = 1/C \quad , \quad \hat{M}_{\mu\nu} = -\hat{M}_{\nu\mu} \tag{6.28}$$

We also have a relationship between $\hat{M}$ and $\hat{p}$:

$$2\hat{x}^\mu\hat{M}_{\mu\nu} = \hat{p}_\nu/C - A\hat{x}_\nu \tag{6.29}$$

We then see that the $\hat{M}$'s generate $\mathbb{SO}\left(p+1,q\right)$ in the case of $C>0$ because $sign\left(\eta\right) = \left(p+1,q\right)$. Similarly the $\hat{M}$'s generate $\mathbb{SO}\left(p,q+1\right)$ in the case of $C<0$ because $sign\left(\eta\right) = \left(p,q+1\right)$. We expected to see these groups in the group of observables because they are the symmetry groups for hyperboloids defined by $x^\mu x_\mu = 1/C$.

So for $T^*M_{C_{p,q}}$ we now have our map σ^{-1}:

$$\begin{aligned}
f\left(x,M\right) &= \sum_{(l)(m)} \tilde{f}^{\nu_1\cdots\nu_{2m}}_{(l)(m)\mu_1\cdots\mu_l} x^{\mu_1}\cdots x^{\mu_l} M_{\nu_1\nu_2}\cdots M_{\nu_{(2m-1)}\nu_{2m}} \\
\overset{\sigma}{\leftrightarrow} \hat{f}\left(\hat{x},\hat{M}\right) &= \sum_{(l)(m)} \tilde{f}^{\nu_1\cdots\nu_{2m}}_{(l)(m)\mu_1\cdots\mu_l} SYM\left(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_l}\hat{M}_{\nu_1\nu_2}\cdots\hat{M}_{\nu_{(2m-1)}\nu_{2m}}\right)
\end{aligned}$$

Again, as before $\hat{f}$ is Hermitian iff $\hat{x}$ and $\hat{M}$ are.

6.8 THE OBSERVABLE ALGEBRA IS THE PSEUDO-ORTHOGONAL GROUP

Now that we have a basis of the observable algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$, we want analyze the Lie group associated to the Lie algebra relations in (6.27). It turns out that the algebra $\widetilde{\mathcal{A}}_{D,\hat{D}}$ is pseudo-orthogonal group $\mathbb{SO}\left(p+1,q+1\right)$:

$$\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C^{\infty}_{A}\left(T^{*}M_{C_{p,q}}\right)\right) = \mathbb{SO}\left(p+1,q+1\right)$$

The way in which this can be seen is by replacing the $\hat{x}$ in the basis $\left(\hat{x},\hat{M}\right)$ for $\widetilde{\mathcal{A}}_{D,\hat{D}}$ by:

$$\hat{p}_{\mu'} = \frac{1}{2\sqrt{|C|}}\left(C\hat{x}^{\nu}\hat{M}_{\nu\mu'} + \frac{CA}{i\hbar}\hat{x}_{\mu'}\right)$$

We organize this into $\hat{M}_{\mu'\nu'}$ where we make a convention that the primed indices (e.g. μ') run from 1 to $(n+2)$ instead of just 1 to $(n+1)$:

$$\hat{M}_{\mu'\nu'} = -\hat{M}_{\nu'\mu'}$$

$$\hat{M}_{(n+2)\mu'} = -\hat{M}_{\mu'(n+2)} = \frac{1}{2\sqrt{|C|}}\hat{p}_{\mu'} = \frac{1}{2\sqrt{|C|}}\left(C\hat{x}^{\nu}\hat{M}_{\nu\mu'} + \frac{CA}{i\hbar}\hat{x}_{\mu'}\right) \text{ for } \mu' = 1,\ldots,n+1$$

and we also define:

$$\eta_{(n+2)(n+2)} = -C/\left|C\right|$$

$$\eta_{(n+2)\mu'} = 0 \ \text{ for } \mu' \neq n+2$$

In appendix C.5 we compute the commutation relation:

$$\left[\hat{M}_{\mu'\nu'},\hat{M}_{\rho'\sigma'}\right] = i\hbar\left(\hat{M}_{\rho'[\mu'}\eta_{\nu']\sigma'} - \hat{M}_{\sigma'[\mu'}\eta_{\nu']\rho'}\right) \tag{6.30}$$

which are then equivalent to (6.28). Thus the $\hat{M}'$'s (i.e., the $\hat{M}_{\mu'\nu'}$'s) form the Lie Algebra of $\mathbb{SO}\left(p+1,q+1\right)$, $\mathfrak{so}\left(p+1,q+1\right)$ for both $C>0$ and $C<0$!

The Summary of the Results:

We now have the following scheme worked out exactly:

- For the configuration space $M_{C_{p,q}}$ with $sign\,(g) = (p,q)$ and $C > 0$:
  - $sign\,(\eta) = (p+1,q)$, $\hat{M}$ generates $\mathbb{SO}\,(p+1,q)$.
  - $sign\,(\eta') = (p+1,q+1)$, $\hat{M}' = \left(\hat{M},\hat{x}\right)$ generates $\widetilde{\mathcal{A}}_{D,\hat{D}} = \mathbb{SO}\,(p+1,q+1)$.
- For the configuration space $M_{C_{p,q}}$ with $sign\,(g) = (p,q)$ and $C < 0$:
  - $sign\,(\eta) = (p,q+1)$, $\hat{M}$ generates $\mathbb{SO}\,(p,q+1)$.
  - $sign\,(\eta') = (p+1,q+1)$, $\hat{M}' = \left(\hat{M},\hat{x}\right)$ generates $\widetilde{\mathcal{A}}_{D,\hat{D}} = \mathbb{SO}\,(p+1,q+1)$.

For example, in the case of dS (AdS) the $\hat{M}$'s generate $\mathbb{SO}\,(1,4)$ ($\mathbb{SO}\,(2,3)$) because for dS $\eta = diag\,(1,-1,-1,-1,-1)$, $C < 0$ and for AdS $\eta = diag\,(1,1,-1,-1,-1)$, $C > 0$. The full algebra of observables $\widetilde{\mathcal{A}}_{D,\hat{D}} = \sigma^{-1}\left(C^{\infty}_{A}\left(T^{*}M_{C_{p,q}}\right)\right) = \mathbb{SO}\,(2,4)$ for *both* dS and AdS.

As we know, $\mathbb{SO}\,(2,4)$ is the conformal group for these space-times, however to assert the claim that this $\mathbb{SO}\,(2,4)$ is the conformal group we need a clear interpretation of the generators.

Additionally, using the equation $\hat{M}_{\mu\nu} = \hat{x}_{[\mu}\hat{p}_{\nu]}$ we compute directly:

$$\hat{M}'^{2} = \hat{M}_{\mu'\nu'}\hat{M}^{\mu'\nu'} = -\frac{1}{2}\left(A - i\hbar n\right)A \tag{6.31}$$

$$\hat{M}^{2} = \hat{M}_{\mu\nu}\hat{M}^{\mu\nu} = \frac{1}{2C}\hat{p}_{\mu}\hat{p}^{\mu} + \hat{M}'^{2} \tag{6.32}$$

6.9 THE KLEIN-GORDON EQUATION FOR DS AND ADS

Now that we have a basis for the algebra of observables $\widetilde{\mathcal{A}}_{D,\hat{D}} := \sigma^{-1}\left(C_{A}^{\infty}\left(T^{*}M_{C_{p,q}}\right)\right) = \mathbb{SO}\left(p+1,q+1\right)$, we can quantize the Hamiltonian for geodesic motion in (3.15). As illustrated in section 3.4, we can now replace $(\hat{x}^{\mu},\hat{p}_{\mu})$ (or equivalently $\left(\hat{x}^{\mu},\hat{M}_{\mu\nu}\right)$) by the pair (x^{μ},T_{μ}) (or equivalently $(x^{\mu},T_{\mu\nu})$) where T_{μ} $(T_{\mu\nu})$ is a differential operator defined by the Lie algebra relations in (6.25) or (6.27).

First start with the Hamiltonian for geodesic motion in (3.15):

$$H = g^{\mu\nu}\left(x\right)p_{\mu}p_{\nu} - m^{2} + \xi R\left(x\right)$$

Before the quantization we calculate $R\left(x\right) = -16C$ and $g^{\mu\nu}\left(x\right)p_{\mu}p_{\nu} = \eta^{\mu\nu}p_{\mu}p_{\nu}$ so that:

$$H = \eta^{\mu\nu}p_{\mu}p_{\nu} - m^{2} - 16\xi C$$

and now we can apply our quantization map σ^{-1}:

$$\hat{H} = \eta^{\mu\nu}\hat{p}_{\mu}\hat{p}_{\nu} - m^{2} - 16\xi C$$

We can express $\hat{p}_{\mu}\hat{p}^{\mu}$ in terms of $\hat{M}$ and $\hat{x}$ using the formula in (6.29) and by substituting into $\hat{H}$:

$$\hat{H} = 2C\hat{M}_{\mu\nu}\hat{M}^{\mu\nu} + \left(A+4i\hbar\right)AC - 16\xi C - m^{2} \tag{6.33}$$

where $\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}\left(1,4\right)$ or $\mathbb{SO}\left(2,3\right)$ for dS or AdS respectively.

Using this as a constraint on the set of allowed physical states we have the equation:

$$\left(2C\hat{M}_{\mu\nu}\hat{M}^{\mu\nu} + \chi C - m^{2}\right)\left|\phi\right\rangle = 0 \tag{6.34}$$

where $\left\langle\phi|\phi\right\rangle = 1$, $\mathbb{C} \ni \chi = \left(A+4i\hbar\right)A - 16\xi$ is an arbitrary constant, and we regard all groups to be in a standard irreducible representation on the set of linear Hilbert space operators.

These subgroups are the symmetry groups of the manifolds for dS or AdS respectively and again, $\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}\left(1,4\right)$ or $\mathbb{SO}\left(2,3\right)$ for dS or

AdS respectively. This is the same equation formulated in Frønsdal C. (1965, 1973, 1975a, 1975b), a well-known result.

Ignoring spin and other complications, the operator $\hat{M}^2$ becomes the Laplace-Beltrami operator $\nabla_\mu \nabla^\mu$ and $\hat{x}^\mu \hat{p}_\mu \to -i\hbar x^\mu \nabla_\mu$ so let $\phi(x) := \langle x|\phi\rangle$ then:

$$\left(2i\hbar C \nabla_\mu \nabla^\mu - \chi C - m^2\right) \phi(x) = 0$$

where $-i\hbar x^\mu \nabla_\mu \phi = A\phi$ (see section 3.4). This equation is the free modified wave equation on AdS that is studied in Frønsdal C. (1973) and therefore the results given here are consistent with what has been done previously.

6.10 THE KLEIN-GORDON EQUATION

This section is a straightforward generalization of the last section. First we will quantize the Hamiltonian for geodesic motion in (3.15), then as in section 3.4 we can find the differential equation that is the Klein-Gordon equation.

First start with the Hamiltonian for geodesic motion in (3.15):

$$H = g^{\mu\nu}(x) p_\mu p_\nu - m^2 + \xi R(x)$$

Before the quantization we calculate $R(x) = -n^2 C$ ($\dim M_{C_{p,q}} = n$) and $g^{\mu\nu}(x) p_\mu p_\nu = \eta^{\mu\nu} p_\mu p_\nu$ so that:

$$H = \eta^{\mu\nu} p_\mu p_\nu - m^2 - n^2 \xi C$$

and now we can apply our quantization map σ^{-1}:

$$\hat{H} = \eta^{\mu\nu} \hat{p}_\mu \hat{p}_\nu - m^2 - n^2 \xi C$$

We can express $\hat{p}_\mu \hat{p}^\mu$ in terms of $\hat{M}$ and $\hat{x}$ using the formula in (6.29) and by substituting into $\hat{H}$:

$$\hat{H} = 2C \hat{M}_{\mu\nu} \hat{M}^{\mu\nu} + (A + ni\hbar) AC - n^2 \xi C - m^2 \tag{6.35}$$

where $\hat{M}_{\mu\nu} \hat{M}^{\mu\nu}$ is a Casimir invariant of the subgroup $\mathbb{SO}(p, q+1)$ or $\mathbb{SO}(p+1, q)$ for $C > 0$ or $C < 0$ respectively.

Using this as a constraint on the set of allowed physical states we have the equation:

$$\left(2C\hat{M}_{\mu\nu}\hat{M}^{\mu\nu} + \chi C - m^2\right)|\phi\rangle = 0 \tag{6.36}$$

where $\langle\phi|\phi\rangle = 1$, $\mathbb{C} \ni \chi = (A + ni\hbar)\,A - n^2\xi$ is an arbitrary constant, and we regard all groups to be in a standard irreducible representation on the set of linear Hilbert space operators.

Ignoring spin as well as other complications, the operator $\hat{M}^2$ becomes the Laplace-Beltrami operator $\nabla_\mu\nabla^\mu$ on $M_{C_{p,q}}$ and $\hat{x}^\mu\hat{p}_\mu \to -i\hbar x^\mu\nabla_\mu$ so let $\phi(x) := \langle x|\phi\rangle$ then:

$$\left(2i\hbar C\nabla_\mu\nabla^\mu - \chi C - m^2\right)\phi(x) = 0$$

where $-i\hbar x^\mu\nabla_\mu\phi = A\phi$ (see section 3.4). This equation is the free modified wave equation on $M_{C_{p,q}}$.

7.0 CONCLUSIONS

In conclusion, the results of this thesis confirm the well known results for the Klein-Gordon equation in Frønsdal C. (1965, 1973, 1975a, 1975b) as well as many others in the case of dS/AdS. The difference is that we confirmed these results in the context of DQ and using the Fedosov star-product. The beautiful thing about these computations is that they are coordinate-free object, algorithmic, and they can be done in principle for any manifold. This is in contrast to some previous techniques in quantization which relied heavily on the symmetries of these particular manifolds. The quantum algebra for all constant curvature manifolds of codimension one was the group $\mathbb{SO}\,(q+1,p+1)$. This algebra contained the symmetry group of the manifold $\mathbb{SO}\,(q,p+1)$ or $\mathbb{SO}\,(q+1,p)$ which plays a fundamental role in the Klein-Gordon equation.

Another result of this thesis is that Hilbert space quantization of any symplectic manifold can be obtained by Fedosov's algorithm without ever needing to go to Fedosov's star-product. As was stated before, Hilbert space quantization may be viewed as a Fedosov construction. This realization was a result of the following observation: whether the bundle over phase-space is of Groenewold-Moyal type or Hilbert space type is irrelevant—as long as the commutators and the action of the phase-space connection remain the same the procedure will work. The last result is a further refinement in the case of cotangent bundles of the formula in (3.8) to the ansatz (5.14) subject to the condition in equation (5.15).

Subjects of future study include the solutions to the Klein-Gordon and Dirac equation using Fedosov quantization for these manifolds, and physical models of particles on these space-times in the context of DQ. The perturbative nature of DQ will ensure that some result will be obtained. Other ideas that will be explored are the connections between DQ, the local theory of quantum mechanics known as the Algebraic Quantum Field Theory, and quantum thermodynamics.

APPENDIX A

NOTES ON THE KLEIN-GORDON EQUATION

The Fedosov quantization map σ^{-1} is used to describe some physical system. For example, in this thesis it is applied to the Hamiltonian for geodesic motion of a single free particle in section 3.4. We regard this merely as a test to see how the procedure will work, and not the final theory. In the case of Minkowski space, the map gives the well-known Klein-Gordon equation and, in the general case, it will suffer many of the same problems. One problem is that the space of all solutions to the Klein-Gordon equation will have some negative probability solutions, and the other problem is the absence of an observer independent definition of a particle. These problems may be solved by using the Dirac equation and using quantum field theory.

To get rid of these negative probabilities in Minkowski space you can use the Dirac equation. Unlike the Klein-Gordon equation, the Dirac equation is a differential equation first order in derivatives thereby forbidding these negative probability solutions.

In GR, including in Minkowski space, even the very meaning of particles is ambiguous. In a standard procedure, as in Woodhouse N. 1980, in GR you take the space of all solutions to the Klein-Gordon equation and to separate positive and negative frequency solutions by introducing a complex structure. It is known in Wald R. 1994 that the choice of this structure is inherently observer dependent, meaning that each observer will disagree, in general, on which states are of positive frequencies and which are negative. The consequence is that, in general, each observer will disagree on the very definition of what a particle is. This ambiguity is observed in Minkowski space in the Unruh effect, where an inertial observer sees no particles but an accelerated observer sees many.

The Unruh effect shows explicitly that you can never, for all observers, say that there is one particle in space-time. This is why the very notion of a single particle quantum theory for all observers is nonsensical in a relativistic setting. To get a coherent quantum theory you need to introduce a quantum field theory (QFT). This involves a process called second quantization—which we are not going to do in this thesis.

With all of these arguments aside, what we are mainly interested in is the process of quantization itself. We are not going to formulate the Dirac equation or make a quantum field theory—we want to see how one, in general, goes about formulating a quantum theory of a single free relativistic particle using Fedosov quantization. This is only the first, but important step to a QFT on curved space-times.

APPENDIX B

THE TWO-SPHERE CASE

B.1 SPHERE IDENTITIES

Useful identities:

$$r = -\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right) + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{s}\cdot\underline{\theta} \text{ for } \{h=0, f=-1/3, g=1\} \tag{B.1}$$

$$\underline{z}\times\underline{x} = \underline{p}\times\underline{x} - \underline{k} \tag{B.2}$$

$$[s^{\mu}, k_{\nu}] = -i\hbar\left(\delta^{\mu}_{\nu} - x^{\mu}x_{\nu}\right) \tag{B.3}$$

$$\underline{z} = \underline{p} - \underline{x}\times\underline{k} \tag{B.4}$$

$$d\underline{p} = \underline{\alpha}\times\underline{x} - \underline{p}\times\underline{\theta} \tag{B.5}$$

$$\theta^{\mu}\theta^{\nu} = \theta^{[\mu}\theta^{\nu]} = \frac{1}{2}\varepsilon^{\mu\nu\rho}\left(\underline{\theta}\times\underline{\theta}\right)_{\rho} = \tilde{\omega}\varepsilon^{\mu\nu\rho}x_{\rho} \tag{B.6}$$

$$\left(\underline{v}\times\underline{w}\right)\times\underline{u} = \delta_{\mu\nu}v^{\mu}\underline{w}u^{\nu} - \underline{v}\left(\underline{w}\cdot\underline{u}\right) \tag{B.7}$$

$$\underline{v} \times (\underline{w} \times \underline{u}) = \delta_{\mu\nu} v^{\mu} \underline{w} u^{\nu} - (\underline{v} \cdot \underline{w})\, \underline{u} \tag{B.8}$$

for all 3-dimensional vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ assuming nothing about $[v_\mu, w_\nu]$, $[v_\mu, u_\nu]$ or $[w_\mu, u_\nu]$.

$$(\underline{v} \cdot \underline{\theta})\, (\underline{x} \times \underline{w}) \cdot \underline{\theta} = \tilde{\omega}\, (\underline{v} \cdot \underline{w}) \tag{B.9}$$

for all 3-dimensional vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ assuming $[\theta^\mu, v_\nu] = [\theta^\mu, w_\nu] = 0$ and assuming nothing about $[v_\mu, w_\nu]$.

Expanding a vector in the basis $\{\underline{x}, \underline{s}, \underline{x} \times \underline{s}\}$ (note $\underline{x} \cdot \underline{s} = 0$ and assume $\underline{s} \neq 0$):

$$\begin{aligned} \underline{u} &= (\underline{u} \cdot \underline{x})\, \underline{x} + \underline{u} \cdot (\underline{x} \times \underline{s}) \left(\frac{1}{s^2}\right) \underline{x} \times \underline{s} + (\underline{u} \cdot \underline{s}) \left(\frac{1}{s^2}\right) \underline{s} \\ &= \underline{x}\, (\underline{x} \cdot \underline{u}) + \underline{x} \times \underline{s} \left(\frac{1}{s^2}\right) (\underline{x} \times \underline{s}) \cdot \underline{u} + \left(\frac{1}{s^2}\right) \underline{s}\, (\underline{s} \cdot \underline{u}) \end{aligned} \tag{B.10}$$

for all 3-dimensional vectors $\underline{u}$ assuming nothing about $[u_\mu, s^\nu]$.

For two vectors u_μ and w_μ such that $\underline{v} \cdot \underline{x} = \underline{w} \cdot \underline{x} = 0$ we have the identities:

$$\underline{v} \times \underline{w} = ((\underline{v} \times \underline{w}) \cdot \underline{x})\, \underline{x} \sim \underline{x} \tag{B.11}$$

$$\underline{z} \cdot (\underline{x} \times \underline{s}) = \underline{p} \cdot (\underline{x} \times \underline{s}) - t \tag{B.12}$$

$$\tilde{D} := D - \hat{D} \text{ for } f = -1/3, g = 1, h = 0 \tag{B.13}$$

$$\tilde{D}\underline{s} = \underline{\theta} \times \underline{s} - \underline{\theta} - \underline{s}\, (\underline{s} \cdot \underline{\theta}) \tag{B.14}$$

$$\tilde{D}\underline{x} = D\underline{x} = \underline{\theta} \times \underline{x} = \frac{1}{s^2} \left(((\underline{x} \times \underline{s}) \cdot \underline{\theta})\, \underline{s} - (\underline{s} \cdot \underline{\theta})\, \underline{x} \times \underline{s} \right. \tag{B.15}$$

$$\tilde{D}\underline{k} = \underline{\theta} \times \underline{k} - \underline{\alpha} - \underline{z} \times \underline{x}\, (\underline{s} \cdot \underline{\theta}) - \underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{\theta} \tag{B.16}$$

$$\tilde{D}\underline{z} = \underline{\theta} \times \underline{z} + \underline{z}\, (\underline{s} \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{x} \times \underline{\theta} \tag{B.17}$$

$$\tilde{D}s^2 = 2\underline{s}\cdot\tilde{D}\underline{s} = -2\left(s^2+1\right)(\underline{s}\cdot\underline{\theta}) \tag{B.18}$$

For an arbitrary function $f\left(s^2\right)$ of s^2:

$$\tilde{D}f = -2f'\left(s^2+1\right)(\underline{s}\cdot\underline{\theta}) \tag{B.19}$$

where $f' := \partial f/\partial\left(s^2\right)$.

In Appendix B.3 there is a derivation of identities (B.14) through (B.18) except (B.15).

EXTRA IDENTS:

$$\left[s^2, (\underline{x}\times\underline{k})\cdot\underline{s}\right] = 0$$

$$s_a f(\underline{k}\cdot\underline{s}) = f(\underline{k}\cdot\underline{s}+1)\, s_a$$

$$r_0 = \frac{1}{3}\left((\underline{k}\cdot\underline{\theta})\, s^2 - \underline{k}\cdot\underline{s}\,(\underline{s}\cdot\underline{\theta})\right)$$

$$[r_0, \underline{s}] = \frac{1}{3}\left((\underline{s}\cdot\underline{\theta})\,\underline{s} - s^2\underline{\theta}\right)$$

$$[r_0, (\underline{s}\cdot\underline{\theta})] = 0$$

$$\left[r_0, s^2\right] = 0 = \left[\underline{z}\cdot\underline{s}, s^2\right]$$

$$[r_0, \underline{k}] = \frac{1}{3}\left(2\underline{s}\,(\underline{k}\cdot\underline{\theta}) - \underline{\theta}t - (\underline{s}\cdot\underline{\theta})\,\underline{k}\right)$$

$$[r_0, \underline{z}] = \frac{1}{3}\left((\underline{s}\cdot\underline{\theta})\,\underline{x}\times\underline{k} - \underline{\theta}\times\underline{x}t - 2\underline{x}\times\underline{s}\,(\underline{k}\cdot\underline{\theta})\right)$$

B.2 PROOF OF (4.12) AND (4.13)

This appendix contains the proof that (4.12) and (4.13) solve the condition (4.10) (which came from (5.11)) in the ansatz (4.11):

Starting with the ansatz in (4.11):

$$r = r_0 + f\left(s^2\right)\underline{z}\cdot\underline{s}\,(\underline{x}\times\underline{s})\cdot\underline{\theta} + g\left(s^2\right)\underline{z}\cdot(\underline{x}\times\underline{s})\,\underline{s}\cdot\underline{\theta} + h\left(s^2\right)\underline{s}\cdot\underline{\theta}$$

where $\underline{z} = \underline{p} - \underline{x}\times\underline{k}$ and $r_0 = \frac{1}{3}\left((\underline{k}\cdot\underline{\theta})\, s^2 - \underline{k}\cdot\underline{s}\,(\underline{s}\cdot\underline{\theta})\right)$.

For this section of the appendix $t = \underline{k} \cdot \underline{s}$, $u = \underline{s} \cdot \underline{s}$, $f' := \partial f / \partial u$ also we used some identities in Appendix B.1.

▶ $\hat{d}r$

$$\begin{aligned}
\hat{d}r &= -\Omega + \hat{d}f\underline{z} \cdot \underline{s} \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + f \left(- \left(\underline{x} \times \underline{\alpha} \right) \cdot \underline{s} + \underline{z} \cdot \underline{\theta} \right) \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + f\underline{z} \cdot \underline{s} \left(\underline{x} \times \underline{\theta} \right) \cdot \underline{\theta} \\
&\quad + \hat{d}g\underline{z} \cdot \left(\underline{x} \times \underline{s} \right) \underline{s} \cdot \underline{\theta} - g \left(\underline{x} \times \underline{\alpha} \right) \cdot \left(\underline{x} \times \underline{s} \right) \underline{s} \cdot \underline{\theta} + g\underline{z} \cdot \left(\underline{x} \times \underline{\theta} \right) \underline{s} \cdot \underline{\theta} \\
&\quad + \hat{d} \left(hs^2 \right) \left(\underline{z} \times \underline{x} \right) \cdot \underline{\theta} - hs^2 \underline{\alpha} \cdot \underline{\theta} \\
&= -\Omega + 2f' \left(\underline{s} \cdot \underline{\theta} \right) \underline{z} \cdot \underline{s} \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + f \left(\underline{\alpha} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + \underline{z} \cdot \underline{s} \tilde{\omega} \right) + 2f\underline{z} \cdot \underline{s} \tilde{\omega} \\
&\quad + 2g' \left(\underline{s} \cdot \underline{\theta} \right) \left(\underline{p} \cdot \left(\underline{x} \times \underline{s} \right) - t \right) \left(\underline{s} \cdot \underline{\theta} \right) + g \left(- \left(\underline{s} \cdot \underline{\alpha} \right) \underline{s} \cdot \underline{\theta} + \underline{z} \cdot \underline{s} \tilde{\omega} \right) \\
&\quad + \left(2h + 2h' s^2 \right) \left(\underline{s} \cdot \underline{\theta} \right) \left(\underline{z} \times \underline{x} \right) \cdot \underline{\theta} - hs^2 \omega \\
&= -\Omega + 2f' \left(\underline{s} \cdot \underline{\theta} \right) \underline{z} \cdot \underline{s} \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + f \left(\underline{\alpha} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + 3\underline{z} \cdot \underline{s} \tilde{\omega} \right) \\
&\quad + 2g' \left(\underline{p} \cdot \left(\underline{x} \times \underline{s} \right) - t - 2 \right) \left(\underline{s} \cdot \underline{\theta} \right)^2 - g \left(\underline{s} \cdot \underline{\alpha} \right) \underline{s} \cdot \underline{\theta} + g\underline{z} \cdot \underline{s} \tilde{\omega} \\
&\quad - \left(2h + 2h' s^2 \right) \underline{z} \cdot \underline{s} \tilde{\omega} - hs^2 \omega
\end{aligned}$$

$$\begin{aligned}
\hat{d}r &= -\Omega + 2f' \left(\left(\underline{z} \cdot \underline{s} \right) \underline{s} \cdot \underline{\theta} + \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} \right) \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} + \left(3f + g \right) \underline{z} \cdot \underline{s} \tilde{\omega} \\
&\quad - g \left(\underline{s} \cdot \underline{\alpha} \right) \underline{s} \cdot \underline{\theta} + f \underline{\alpha} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} \\
&\quad - \left(2h + 2h' s^2 \right) \underline{z} \cdot \underline{s} \tilde{\omega} - hs^2 \omega \\
\hat{d}r &= -\Omega + \left(2f' s^2 + 3f + g - 2h - 2h' s^2 \right) \underline{z} \cdot \underline{s} \tilde{\omega} - g \left(\underline{s} \cdot \underline{\alpha} \right) \underline{s} \cdot \underline{\theta} \\
&\quad + f \underline{\alpha} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} - hs^2 \omega
\end{aligned}$$

▶ Dr

$$\begin{aligned}
Dr &= \frac{1}{3} D \left(\left(\underline{s} \cdot \underline{\theta} \right) t - s^2 \left(\underline{k} \cdot \underline{\theta} \right) \right) + fD \left(\underline{z} \cdot \underline{s} \left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} \right) + gD \left(\left(\underline{p} \cdot \left(\underline{x} \times \underline{s} \right) - t \right) \underline{s} \cdot \underline{\theta} \right) \\
&\quad + D \left(hs^2 \left(\underline{p} \times \underline{x} - \underline{k} \right) \cdot \underline{\theta} \right)
\end{aligned}$$

Some useful identities are,

$$D\underline{s} = \underline{\theta} \times \underline{s}$$

$$D\underline{k} = \underline{\theta} \times \underline{k} - \frac{2}{3}\underline{\theta} \times \underline{x}\left(\underline{p} \cdot \underline{s}\right) + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{s} \times \underline{x}\right)$$

$$Ds^2 = 0$$

$$D\left(\underline{s} \cdot \underline{\theta}\right) = 0$$

$$D\left(\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta}\right) = \left(\left(\underline{\theta} \times \underline{x}\right) \times \underline{s}\right) \cdot \underline{\theta} + \left(\underline{x} \times \left(\underline{\theta} \times \underline{s}\right)\right) \cdot \underline{\theta} + \left(\underline{x} \times \underline{s}\right) \cdot \left(\underline{\theta} \times \underline{\theta}\right) = 0$$

$$\begin{aligned}
D\left(\underline{k} \cdot \underline{\theta}\right) &= \left(\underline{\theta} \times \underline{k} - \frac{2}{3}\left(\underline{p} \cdot \underline{s}\right)\underline{\theta} \times \underline{x} + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\underline{s} \times \underline{x}\right) \cdot \underline{\theta} + \underline{k} \cdot \left(\underline{\theta} \times \underline{\theta}\right) \\
&= -\frac{2}{3}\left(\underline{p} \cdot \underline{s}\right)\left(\underline{\theta} \times \underline{x}\right) \cdot \underline{\theta} + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{s} \times \underline{x}\right) \cdot \underline{\theta} \\
&= \frac{4}{3}\left(\underline{p} \cdot \underline{s}\right)\tilde{\omega} - \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta} \\
D\left(\underline{k} \cdot \underline{\theta}\right) &= \frac{4}{3}\left(\underline{p} \cdot \underline{s}\right)\tilde{\omega} - \frac{1}{3}\left(\underline{p} \cdot \underline{s}\right)\tilde{\omega} = \tilde{\omega}\left(\underline{p} \cdot \underline{s}\right)
\end{aligned}$$

$$\begin{aligned}
D\left(t\left(\underline{s} \cdot \underline{\theta}\right)\right) &= \left(\left(\underline{\theta} \times \underline{k} - \frac{2}{3}\left(\underline{p} \cdot \underline{s}\right)\underline{\theta} \times \underline{x} + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\underline{s} \times \underline{x}\right) \cdot \underline{s} + \underline{k} \cdot \left(\underline{\theta} \times \underline{s}\right)\right)\left(\underline{s} \cdot \underline{\theta}\right) \\
D\left(t\left(\underline{s} \cdot \underline{\theta}\right)\right) &= -\frac{2}{3}\left(\underline{p} \cdot \underline{s}\right)\left(\underline{\theta} \times \underline{x}\right) \cdot \underline{s}\left(\underline{s} \cdot \underline{\theta}\right) = \frac{2\tilde{\omega}}{3}\left(\underline{p} \cdot \underline{s}\right)s^2
\end{aligned}$$

$$\begin{aligned}
D\left(\underline{s} \cdot \underline{\alpha}\right) &= \left(\underline{\theta} \times \underline{s}\right) \cdot \underline{\alpha} + \underline{s} \cdot \left(\underline{\theta} \times \underline{\alpha} - \frac{2}{3}\underline{\theta} \times \underline{x}\left(\underline{p} \cdot \underline{\theta}\right) + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\underline{\theta} \times \underline{x}\right) \\
&= -\frac{2}{3}\underline{s} \cdot \left(\underline{\theta} \times \underline{x}\right)\left(\underline{p} \cdot \underline{\theta}\right) + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\underline{s} \cdot \left(\underline{\theta} \times \underline{x}\right) \\
&= \frac{2}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta} + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta} = \tilde{\omega}\left(\underline{p} \cdot \underline{s}\right)
\end{aligned}$$

This is true because,

$$\left[D, \hat{d}\right]\hat{h} = \left(DQ\right)\hat{h} = \left(D\left(\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta}\right)\right)\hat{h} = 0$$

for any $\hat{h}$, so we get:

$$\begin{aligned}
Dr &= \frac{1}{3}\left(1-\frac{2}{3}\right)s^2\tilde{\omega}\left(\underline{p}\cdot\underline{s}\right)+fD\left(\underline{z}\cdot\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} \\
&\quad +gD\left(\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)\right)\underline{s}\cdot\underline{\theta}-\frac{2\tilde{\omega}}{3}g\left(\underline{p}\cdot\underline{s}\right)s^2+hs^2D\left(\left(\underline{p}\times\underline{x}-\underline{k}\right)\cdot\underline{\theta}\right) \\
&= \frac{1}{3}\left(\frac{1}{3}-2g\right)s^2\tilde{\omega}\left(\underline{p}\cdot\underline{s}\right)+(Df)+(Dg)+(Dh)
\end{aligned}$$

$$\begin{aligned}
(Df) &= fD\left(\left(\underline{z}\cdot\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right) \\
&= f\left(\begin{pmatrix} \underline{\alpha}\times\underline{x}-\underline{p}\times\underline{\theta}-\left(\underline{\theta}\times\underline{x}\right)\times\underline{k} \\ -\underline{x}\times\begin{pmatrix} \underline{\theta}\times\underline{k}-\frac{2}{3}\underline{\theta}\times\underline{x}\left(\underline{p}\cdot\underline{s}\right) \\ +\frac{1}{3}\left(\underline{p}\cdot\underline{\theta}\right)\left(\underline{s}\times\underline{x}\right)\end{pmatrix}\end{pmatrix}\cdot\underline{s}+\underline{z}\cdot\left(\underline{\theta}\times\underline{s}\right)\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} \\
&= f\left(\underline{\alpha}\cdot\left(\underline{x}\times\underline{s}\right)+\frac{2}{3}\underline{s}\cdot\underline{\theta}\left(\underline{p}\cdot\underline{s}\right)-\frac{1}{3}\left(\underline{p}\cdot\underline{\theta}\right)s^2\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} \\
&= f\left(\underline{\alpha}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}+\frac{2}{3}s^2\left(\underline{p}\cdot\underline{s}\right)\tilde{\omega}-\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)s^2\tilde{\omega}\right) \\
&= f\left(\underline{\alpha}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}+\frac{1}{3}s^2\left(\underline{p}\cdot\underline{s}\right)\tilde{\omega}\right)
\end{aligned}$$

$$(Dg)=gD\left(\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)\right)\underline{s}\cdot\underline{\theta}=g\left(\left(\underline{\alpha}\times\underline{x}-\underline{p}\times\underline{\theta}\right)\cdot\left(\underline{x}\times\underline{s}\right)+\underline{p}\cdot D\left(\underline{x}\times\underline{s}\right)\right)\underline{s}\cdot\underline{\theta}$$

we calculate,

$$D\left(\underline{x}\times\underline{s}\right)=\left(\underline{\theta}\times\underline{x}\right)\times\underline{s}+\underline{x}\times\left(\underline{\theta}\times\underline{s}\right)=\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}\propto\underline{x}$$

$$\begin{aligned}
(Dg) &= g\left(\left(\underline{\alpha}\times\underline{x}\right)\cdot\left(\underline{x}\times\underline{s}\right)\right)\underline{s}\cdot\underline{\theta} \\
(Dg) &= -g\left(\underline{s}\cdot\underline{\alpha}\right)\underline{s}\cdot\underline{\theta}
\end{aligned}$$

$$\begin{aligned}
(Dh) &= hs^2D\left(\left(\underline{p}\times\underline{x}-\underline{k}\right)\cdot\underline{\theta}\right) \\
&= hs^2\left(-\left(\underline{x}\times d\underline{p}\right)\cdot\underline{\theta}-D\left(\underline{k}\cdot\underline{\theta}\right)\right) \\
&= -hs^2\omega-hs^2\underline{p}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

Putting back into Dr,

$$\begin{aligned}
Dr &= \left(\frac{1}{9}-\frac{2g}{3}+\frac{f}{3}-h\right)s^2\tilde{\omega}\left(\underline{p}\cdot\underline{s}\right) \\
&\quad +f\underline{\alpha}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}-g\left(\underline{s}\cdot\underline{\alpha}\right)\underline{s}\cdot\underline{\theta}-hs^2\omega
\end{aligned}$$

We now compute $\Omega - Dr + \hat{d}r$,

$$\begin{aligned}
\Omega - Dr + \hat{d}r &= \left(-\frac{1}{9} + \frac{2g}{3} - \frac{f}{3} + h\right) s^2\tilde{\omega}\left(\underline{p}\cdot\underline{s}\right) - f\underline{\alpha}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} \\
&\quad +g\left(\underline{s}\cdot\underline{\alpha}\right)\underline{s}\cdot\underline{\theta} + hs^2\omega + \left(2f's^2 + 3f + g - 2h - 2h's^2\right)\underline{z}\cdot\underline{s}\tilde{\omega} \\
&\quad -g\left(\underline{s}\cdot\underline{\alpha}\right)\underline{s}\cdot\underline{\theta} + f\underline{\alpha}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} - hs^2\omega \\
&= \left(-\frac{1}{9} + \frac{2g}{3} - \frac{f}{3} + h\right) s^2\tilde{\omega}\left(\underline{p}\cdot\underline{s}\right) \\
&\quad + \left(\left(2f' - 2h'\right)s^2 + 3f + g - 2h\right)\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

Now we compute r^2,

$$r^2 = r_1^2 + [r_1, r_2] + r_2^2$$

where,

$$r_1^2 = r_0^2 + [r_0, r_1'] + r_1'^2$$

$$r_1 = r_0 + f\underline{z}\cdot\underline{s}\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} + g\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{s}\cdot\underline{\theta}$$

$$r' = f\underline{z}\cdot\underline{s}\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} + g\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{s}\cdot\underline{\theta}$$

$$r_2 = hs^2\left(\underline{z}\times\underline{x}\right)\cdot\underline{\theta}$$

$\underline{r_1^2 = r_0^2 + [r_0, r_1'] + r_1'^2}$

r_0^2,

$$\begin{aligned}
18r_0^2 &= 9\left[r_0, r_0\right] = 3\left[r_0, \left(\underline{s}\cdot\underline{\theta}\right)t - s^2\left(\underline{k}\cdot\underline{\theta}\right)\right] \\
&= 3s^2\left[r_0, \left(\underline{k}\cdot\underline{\theta}\right)\right] - 3\left[r_0, \left(\underline{s}\cdot\underline{\theta}\right)t\right] \\
&= 3s^2\left[r_0, \left(\underline{k}\cdot\underline{\theta}\right)\right] + 3\left(\underline{s}\cdot\underline{\theta}\right)\left[r_0, t\right]
\end{aligned}$$

$$3\left[r_0, t\right] = [\underbrace{\left(\underline{s}\cdot\underline{\theta}\right)t - s^2\left(\underline{k}\cdot\underline{\theta}\right)}_{\text{degree 2-1=1}}, t] = 3i\hbar r_0 = i\hbar s^2\left(\underline{k}\cdot\underline{\theta}\right) - i\hbar\left(\underline{s}\cdot\underline{\theta}\right)t$$

$$\begin{aligned}
18r_0^2/\left(i\hbar\right) &= -s^2\left(\underline{\theta}t + \left(\underline{s}\cdot\underline{\theta}\right)\underline{k} - 2\underline{s}\left(\underline{k}\cdot\underline{\theta}\right)\right)\cdot\underline{\theta} \\
&\quad - \left(\underline{s}\cdot\underline{\theta}\right)\left(\left(\underline{s}\cdot\underline{\theta}\right)t - s^2\left(\underline{k}\cdot\underline{\theta}\right)\right) \\
&= s^2\left(\underline{s}\cdot\underline{\theta}\right)\left(\underline{k}\cdot\underline{\theta}\right) - s^2\left(\left(\underline{s}\cdot\underline{\theta}\right)\underline{k}\cdot\underline{\theta} + 2\underline{s}\cdot\underline{\theta}\left(\underline{k}\cdot\underline{\theta}\right)\right) \\
&= -2s^2\left(\underline{s}\cdot\underline{\theta}\right)\underline{k}\cdot\underline{\theta}
\end{aligned}$$

$$\begin{aligned}
18r_0^2/(i\hbar) &= s^2\left(2\underline{s}\left(\underline{k}\cdot\underline{\theta}\right)-\underline{\theta}t-\left(\underline{s}\cdot\underline{\theta}\right)\underline{k}\right)\cdot\underline{\theta}+\left(\underline{s}\cdot\underline{\theta}\right)\left(s^2\left(\underline{k}\cdot\underline{\theta}\right)-\left(\underline{s}\cdot\underline{\theta}\right)t\right)\\
&= s^2\left(\underline{s}\cdot\underline{\theta}\right)\left(\underline{k}\cdot\underline{\theta}\right)-s^2\left(2\underline{s}\cdot\underline{\theta}\left(\underline{k}\cdot\underline{\theta}\right)+\left(\underline{s}\cdot\underline{\theta}\right)\underline{k}\cdot\underline{\theta}\right)\\
&= -2s^2\left(\underline{s}\cdot\underline{\theta}\right)\underline{k}\cdot\underline{\theta}
\end{aligned}$$

$$r_0^2=\frac{i\hbar}{9}s^2\left(\underline{x}\times\underline{k}\right)\cdot\underline{s}\tilde{\omega}$$

$[r_0, r']$,

$$\begin{aligned}
[r_0, r'] &= f\left[r_0, \underline{z}\cdot\underline{s}\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right]+g\left[r_0,\left(\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)-t\right)\underline{s}\cdot\underline{\theta}\right]\\
&= f\left(\left[r_0,\underline{z}\cdot\underline{s}\right]\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}+\underline{z}\cdot\underline{s}\left[r_0,\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right]\right)\\
&\quad +g\left[r_0,\left(\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)-t\right)\right]\underline{s}\cdot\underline{\theta}\\
&= (f1)+(f2)+(g)
\end{aligned}$$

$$\begin{aligned}
3\,(f1)\,/\,(i\hbar) &= 3\left[r_0,\underline{z}\cdot\underline{s}\right]\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\\
&= f\left(s^2\left[\left(\underline{k}\cdot\underline{\theta}\right),\underline{z}\cdot\underline{s}\right]-\left[\left(\underline{s}\cdot\underline{\theta}\right)t,\underline{z}\cdot\underline{s}\right]\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\\
&= f\left(s^2\left[\left(\underline{k}\cdot\underline{\theta}\right),\underline{z}\cdot\underline{s}\right]-\left[\left(\underline{s}\cdot\underline{\theta}\right),\underline{z}\cdot\underline{s}\right]t-\left(\underline{s}\cdot\underline{\theta}\right)\left[t,\underline{z}\cdot\underline{s}\right]\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\\
&= i\hbar f\left(\left(\underline{s}\cdot\underline{\theta}\right)\underline{p}\cdot\underline{s}-\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}t-s^2\underline{z}\cdot\underline{\theta}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\\
&= i\hbar f\left(s^2\underline{z}\cdot\underline{s}\tilde{\omega}-\underline{p}\cdot\underline{s}\tilde{\omega}s^2\right)=i\hbar f s^2\left(\underline{x}\times\underline{k}\right)\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$(f1)\,/\,(i\hbar)=\frac{f}{3}s^2\left(\underline{x}\times\underline{k}\right)\cdot\underline{s}\tilde{\omega}$$

$$\begin{aligned}
(f2)\,/\,(i\hbar) &= f\underline{z}\cdot\underline{s}\left[r_0,\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right]=\frac{f}{3}\underline{z}\cdot\underline{s}\left(\underline{x}\times\left(s^2\underline{\theta}-\left(\underline{s}\cdot\underline{\theta}\right)\underline{s}\right)\right)\cdot\underline{\theta}\\
&= \frac{f}{3}\underline{z}\cdot\underline{s}\left(s^2\underline{x}\times\underline{\theta}-\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\\
&= \frac{f}{3}\underline{z}\cdot\underline{s}\left(\left(2s^2\tilde{\omega}-s^2\tilde{\omega}\right)\right)=\frac{f}{3}\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$\begin{aligned}
(f2)\,/\,(i\hbar) &= f\underline{z}\cdot\underline{s}\left[r_0,\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right]=\frac{f}{3}\underline{z}\cdot\underline{s}\left(\underline{x}\times\left(\left(\underline{s}\cdot\underline{\theta}\right)\underline{s}-s^2\underline{\theta}\right)\right)\cdot\underline{\theta}\\
&= \frac{f}{3}\underline{z}\cdot\underline{s}\left(\left(\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}\times\underline{s}-s^2\underline{x}\times\underline{\theta}\right)\right)\cdot\underline{\theta}\\
&= \frac{f}{3}\underline{z}\cdot\underline{s}\left(\left(s^2\tilde{\omega}-2s^2\tilde{\omega}\right)\right)=-\frac{f}{3}\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$(f2)/(i\hbar) = -\frac{f}{3}s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

$$\begin{aligned}
[r_0,\underline{z}]/(i\hbar) &= -[r_0,\underline{x}\times\underline{k}]/(i\hbar) \\
&= \frac{1}{3}\underline{x}\times(\underline{\theta}t+(\underline{s}\cdot\underline{\theta})\underline{k}-2\underline{s}(\underline{k}\cdot\underline{\theta})) \\
[r_0,\underline{z}]/(i\hbar) &= \frac{1}{3}((\underline{s}\cdot\underline{\theta})\underline{x}\times\underline{k}-\underline{\theta}\times\underline{x}t-2\underline{x}\times\underline{s}(\underline{k}\cdot\underline{\theta}))
\end{aligned}$$

$$\begin{aligned}
3(g)/(i\hbar) &= 3g[r_0,\underline{z}\cdot(\underline{x}\times\underline{s})]\underline{s}\cdot\underline{\theta}/(i\hbar) \\
&= g\begin{pmatrix} ((\underline{s}\cdot\underline{\theta})\underline{x}\times\underline{k}-\underline{\theta}\times\underline{x}t-2\underline{x}\times\underline{s}(\underline{k}\cdot\underline{\theta}))\cdot(\underline{x}\times\underline{s}) \\ +\underline{z}\cdot(\underline{x}\times((\underline{s}\cdot\underline{\theta})\underline{s}-s^2\underline{\theta})) \end{pmatrix}\underline{s}\cdot\underline{\theta} \\
&= g\left(\left((\underline{s}\cdot\underline{\theta})t-(\underline{s}\cdot\underline{\theta})t-2s^2(\underline{k}\cdot\underline{\theta})\right)-\underline{z}\cdot(\underline{x}\times\underline{\theta})s^2\right)\underline{s}\cdot\underline{\theta} \\
&= -g\left(2s^2(\underline{k}\cdot\underline{\theta})+\underline{z}\cdot(\underline{x}\times\underline{\theta})s^2\right)\underline{s}\cdot\underline{\theta} \\
&= g\left(2s^2(\underline{x}\times\underline{k})\cdot\underline{s}\tilde{\omega}-\underline{z}\cdot\underline{s}\tilde{\omega}s^2\right)
\end{aligned}$$

$$[r_0,\underline{s}]/(i\hbar) = \frac{1}{3}\left((\underline{s}\cdot\underline{\theta})\underline{s}-s^2\underline{\theta}\right)$$

$$(g)/(i\hbar) = -\frac{2g}{3}s^2(\underline{x}\times\underline{k})\cdot\underline{s}\tilde{\omega}-\frac{g}{3}s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

$$[r_0,r']/(i\hbar) = \left(\frac{f}{3}-\frac{2g}{3}\right)s^2(\underline{x}\times\underline{k})\cdot\underline{s}\tilde{\omega}-\left(\frac{f}{3}+\frac{g}{3}\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

r'^2,

$$2r'^2 = (ff)+2(fg)+(gg)$$

$$\begin{aligned}
(ff)/(i\hbar) &= [f\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta},f\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta}]/(i\hbar) \\
&= f^2[\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta},\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta}]/(i\hbar) \\
&= f^2/(i\hbar)\begin{pmatrix} [\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta},\underline{z}\cdot\underline{s}](\underline{x}\times\underline{s})\cdot\underline{\theta} \\ +\underline{z}\cdot\underline{s}[\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta},(\underline{x}\times\underline{s})\cdot\underline{\theta}] \end{pmatrix} \\
&= 2f^2/(i\hbar)\,\underline{z}\cdot\underline{s}[(\underline{x}\times\underline{s})\cdot\underline{\theta},\underline{z}\cdot\underline{s}](\underline{x}\times\underline{s})\cdot\underline{\theta} \\
&= 2f^2\underline{z}\cdot\underline{s}(\underline{x}\times(\underline{x}\times\underline{s}))\cdot\underline{\theta}(\underline{x}\times\underline{s})\cdot\underline{\theta} \\
&= -2f^2\underline{z}\cdot\underline{s}(\underline{s}\cdot\underline{\theta})(\underline{x}\times\underline{s})\cdot\underline{\theta} = -2f^2s^2\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$(ff)/(i\hbar) = -2f^2s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

$$
\begin{aligned}
(fg)/(i\hbar) &= [f\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta}, g\underline{z}\cdot(\underline{x}\times\underline{s})\underline{s}\cdot\underline{\theta}]/(i\hbar) \\
&= g/(i\hbar)\begin{pmatrix} [f\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta}, \underline{z}\cdot(\underline{x}\times\underline{s})]\underline{s}\cdot\underline{\theta} \\ +\underline{z}\cdot(\underline{x}\times\underline{s})[f\underline{z}\cdot\underline{s}(\underline{x}\times\underline{s})\cdot\underline{\theta}, \underline{s}\cdot\underline{\theta}] \end{pmatrix} \\
&= g/(i\hbar)\begin{pmatrix} \begin{pmatrix} \left[\underline{z}\cdot\underline{s}, \underline{p}\cdot(\underline{x}\times\underline{s})-t\right] f(\underline{x}\times\underline{s})\cdot\underline{\theta} \\ +\underline{z}\cdot\underline{s}\left[f(\underline{x}\times\underline{s})\cdot\underline{\theta}, \underline{p}\cdot(\underline{x}\times\underline{s})-t\right] \end{pmatrix}\underline{s}\cdot\underline{\theta} \\ -i\hbar\underline{z}\cdot(\underline{x}\times\underline{s}) f((\underline{x}\times\underline{s})\cdot\underline{\theta})^2 \end{pmatrix} \\
&= g/(i\hbar)\left(\begin{pmatrix} \left(\left[\underline{z}\cdot\underline{s}, \underline{p}\cdot(\underline{x}\times\underline{s})\right]-i\hbar\underline{p}\cdot\underline{s}\right) f(\underline{x}\times\underline{s})\cdot\underline{\theta} \\ -\underline{z}\cdot\underline{s}[f(\underline{x}\times\underline{s})\cdot\underline{\theta}, t] \end{pmatrix}\underline{s}\cdot\underline{\theta}\right) \\
&= g\left(\begin{pmatrix} \left(-\underline{p}\cdot(\underline{x}\times(\underline{x}\times\underline{s}))-\underline{p}\cdot\underline{s}\right) f(\underline{x}\times\underline{s})\cdot\underline{\theta} \\ -\underline{z}\cdot\underline{s}([f,t]/(i\hbar)(\underline{x}\times\underline{s})\cdot\underline{\theta}+f(\underline{x}\times\underline{s})\cdot\underline{\theta}) \end{pmatrix}\underline{s}\cdot\underline{\theta}\right) \\
&= g\left(\underline{z}\cdot\underline{s}\left(2f's^2s^2\tilde{\omega}+fs^2\tilde{\omega}\right)\right) = \left(2gf's^2+gf\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}
$$

$$
(fg)/(i\hbar) = \left(2gf's^2+gf\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}
$$

$$
\begin{aligned}
(gg)/(i\hbar) &= \left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\underline{s}\cdot\underline{\theta}, g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\underline{s}\cdot\underline{\theta}\right]/(i\hbar) \\
&= \left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\underline{s}\cdot\underline{\theta}, g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\right]/(i\hbar)\underline{s}\cdot\underline{\theta} \\
&\quad +g\underline{z}\cdot(\underline{x}\times\underline{s})\left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right), \underline{s}\cdot\underline{\theta}\right]/(i\hbar)\underline{s}\cdot\underline{\theta} \\
&= \begin{pmatrix} \left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right), g\underline{p}\cdot(\underline{x}\times\underline{s})\right]\underline{s}\cdot\underline{\theta} \\ -\left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\underline{s}\cdot\underline{\theta}, g\right]t \\ -g\left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\underline{s}\cdot\underline{\theta}, t\right] \end{pmatrix}\underline{s}\cdot\underline{\theta}/(i\hbar) \\
&\quad +g\underline{z}\cdot(\underline{x}\times\underline{s})g\left(-[t,\underline{s}\cdot\underline{\theta}]/(i\hbar)\right)\underline{s}\cdot\underline{\theta} \\
&= \begin{pmatrix} -g\left[t, g\underline{p}\cdot(\underline{x}\times\underline{s})\right]\underline{s}\cdot\underline{\theta}+g[t,g]\underline{s}\cdot\underline{\theta}t \\ -2g\left[g\left(\underline{p}\cdot(\underline{x}\times\underline{s})-t\right)\underline{s}\cdot\underline{\theta}, t\right] \end{pmatrix}\underline{s}\cdot\underline{\theta}/(i\hbar) \\
&\quad +g\underline{z}\cdot(\underline{x}\times\underline{s})g(\underline{s}\cdot\underline{\theta})^2 \\
&= (\sim\underline{s}\cdot\underline{\theta})\underline{s}\cdot\underline{\theta} = 0
\end{aligned}
$$

$$
(gg)/(i\hbar) = 0
$$

$$
r'^2/(i\hbar) = \left(2gf's^2-gf-f^2\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}
$$

We have now r_1^2,

$$[r_0, r'] / (i\hbar) = \left(\frac{f}{3} - \frac{2g}{3}\right) s^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega} - \left(\frac{f}{3} + \frac{g}{3}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}$$

$$\begin{aligned}
r_1^2 / (i\hbar) &= \left(r_0^2 + [r_0, r_1'] + r_1'^2\right) / (i\hbar) \\
&= \frac{1}{9} s^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega} + \left(\left(\frac{f}{3} - \frac{2g}{3}\right) s^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega} - \left(\frac{f}{3} + \frac{g}{3}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}\right) \\
&\quad + \left(2gf's^2 + gf - f^2\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega} \\
&= \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3}\right) s^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega} + \left(2gf's^2 + gf - f^2 - \frac{f}{3} - \frac{g}{3}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega} \\
&= \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} \\
&\quad + \left(2gf's^2 + gf - f^2 - \frac{f}{3} - \frac{g}{3} - \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3}\right)\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
\end{aligned}$$

$$r_1^2 / (i\hbar) = \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2gf's^2 + gf - f^2 - \frac{2f}{3} + \frac{g}{3} - \frac{1}{9}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}$$

$[r_1, r_2]$,

$$[r_1, r_2] = [r_0, r_2] + [r', r_2]$$

$$\begin{aligned}
3[r_0, r_2] / (i\hbar) &= 3hs^2 [r_0, (\underline{z} \times \underline{x}) \cdot \underline{\theta}] / (i\hbar) = -3hs^2 [r_0, \underline{k} \cdot \underline{\theta}] / (i\hbar) \\
&= hs^2 (\underline{\theta} t + (\underline{s} \cdot \underline{\theta}) \underline{k} - 2\underline{s} (\underline{k} \cdot \underline{\theta})) \cdot \underline{\theta} \\
&= hs^2 ((\underline{s} \cdot \underline{\theta}) \underline{k} \cdot \underline{\theta} + 2\underline{s} \cdot \underline{\theta} (\underline{k} \cdot \underline{\theta})) \\
&= 3hs^2 \underline{s} \cdot \underline{\theta} (\underline{k} \cdot \underline{\theta}) = -3hs^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega}
\end{aligned}$$

$$[r_0, r_2] / (i\hbar) = -hs^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega}$$

$$[r', r_2] = \underbrace{\left[f\underline{z} \cdot \underline{s} (\underline{x} \times \underline{s}) \cdot \underline{\theta}, hs^2 (\underline{z} \times \underline{x}) \cdot \underline{\theta}\right]}_{(fh)} + \underbrace{\left[g\underline{z} \cdot (\underline{x} \times \underline{s}) \underline{s} \cdot \underline{\theta}, hs^2 (\underline{z} \times \underline{x}) \cdot \underline{\theta}\right]}_{(gh)}$$

$$\begin{aligned}
(fh) / (i\hbar) &= hs^2 [f\underline{z} \cdot \underline{s} (\underline{x} \times \underline{s}) \cdot \underline{\theta}, (\underline{z} \times \underline{x}) \cdot \underline{\theta}] / (i\hbar) \\
&= hs^2 [f\underline{z} \cdot \underline{s} (\underline{x} \times \underline{s}) \cdot \underline{\theta}, -\underline{k} \cdot \underline{\theta}] / (i\hbar) = hs^2 [-\underline{k} \cdot \underline{\theta}, f\underline{z} \cdot \underline{s} (\underline{x} \times \underline{s}) \cdot \underline{\theta}] / (i\hbar) \\
&= hs^2 (2f' (\underline{s} \cdot \underline{\theta}) \underline{z} \cdot \underline{s} (\underline{x} \times \underline{s}) \cdot \underline{\theta} + f\underline{z} \cdot \underline{\theta} (\underline{x} \times \underline{s}) \cdot \underline{\theta} + f\underline{z} \cdot \underline{s} (\underline{x} \times \underline{\theta}) \cdot \underline{\theta}) \\
&= hs^2 (2f' (\underline{z} \cdot \underline{s} (\underline{s} \cdot \underline{\theta}) + (\underline{x} \times \underline{s}) \cdot \underline{\theta}) (\underline{x} \times \underline{s}) \cdot \underline{\theta} + f\underline{z} \cdot \underline{s}\tilde{\omega} + 2f\underline{z} \cdot \underline{s}\tilde{\omega}) \\
&= hs^2 \left(2f's^2 \underline{z} \cdot \underline{s}\tilde{\omega} + 3f\underline{z} \cdot \underline{s}\tilde{\omega}\right) = \left(2f'hs^2 + 3fh\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
\end{aligned}$$

$$[\underline{z}\cdot\underline{s},\underline{s}]\,/\,(i\hbar) = -\left[(\underline{x}\times\underline{k})\cdot\underline{s},\underline{s}\right]/\,(i\hbar) = \varepsilon_{abc}x^a s^c = -\underline{x}\times\underline{s}$$

$$(fh)\,/\,(i\hbar) = \left(2f'hs^2 + 3fh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

$$\begin{aligned}
(gh)\,/\,(i\hbar) &= \left[g\underline{p}\cdot(\underline{x}\times\underline{s})\,\underline{s}\cdot\underline{\theta}, hs^2\,(\underline{z}\times\underline{x})\cdot\underline{\theta}\right] - \left[gt\underline{s}\cdot\underline{\theta}, hs^2\,(\underline{z}\times\underline{x})\cdot\underline{\theta}\right] \\
&= (1)+(2)
\end{aligned}$$

$$\begin{aligned}
(1) &= -hs^2\left[g\underline{p}\cdot(\underline{x}\times\underline{s})\,\underline{s}\cdot\underline{\theta},\underline{k}\cdot\underline{\theta}\right]/\,(i\hbar) \\
&= hs^2\left(g\underline{p}\cdot(\underline{x}\times\underline{\theta})\,\underline{s}\cdot\underline{\theta} + 2g'\,(\underline{s}\cdot\underline{\theta})\,\underline{p}\cdot(\underline{x}\times\underline{s})\,\underline{s}\cdot\underline{\theta}\right) \\
&= hs^2\left(g\underline{p}\cdot(\underline{x}\times\underline{\theta})\,\underline{s}\cdot\underline{\theta}\right) = hs^2\left(g\underline{s}\cdot\underline{\theta}\left(\underline{x}\times\underline{p}\right)\cdot\underline{\theta}\right) \\
&= hs^2 g\underline{p}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$\begin{aligned}
(2) &= -\left[gt\underline{s}\cdot\underline{\theta}, hs^2\right](\underline{z}\times\underline{x})\cdot\underline{\theta}/\,(i\hbar) - hs^2\left[gt\underline{s}\cdot\underline{\theta},(\underline{z}\times\underline{x})\cdot\underline{\theta}\right]/\,(i\hbar) \\
&= -g/\,(i\hbar)\left[t,hs^2\right]\underline{s}\cdot\underline{\theta}\,(\underline{x}\times\underline{z})\cdot\underline{\theta} - hs^2\left[-\underline{k}\cdot\underline{\theta}, gt\underline{s}\cdot\underline{\theta}\right]/\,(i\hbar) \\
&= g\left(2h's^4 + 2hs^2\right)(-\underline{z}\cdot\underline{s}\tilde{\omega}) - hs^2\left(2g'\underline{s}\cdot\underline{\theta}t\underline{s}\cdot\underline{\theta} + g\underline{k}\cdot\underline{\theta}\underline{s}\cdot\underline{\theta}\right) \\
&= -\left(2gh's^2 + 2gh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega} - ghs^2\,(\underline{x}\times\underline{k})\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$\begin{aligned}
(gh)\,/\,(i\hbar) &= (1)+(2) = -hs^2g\underline{p}\cdot\underline{s}\tilde{\omega} + \left(\left(-2gh's^2 - 2gh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega} - ghs^2\,(\underline{x}\times\underline{k})\cdot\underline{s}\tilde{\omega}\right) \\
&= \left(-2gh's^2 - gh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$(gh)\,/\,(i\hbar) = \left(-2gh's^2 - gh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

$$(fh)\,/\,(i\hbar) = \left(2f'hs^2 + 3fh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}$$

Putting it back into $[r', r_2]$,

$$\begin{aligned}
[r', r_2]\,/\,(i\hbar) &= \left(2f'hs^2 + 3fh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega} + \left(-2gh's^2 - gh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega} \\
[r', r_2]\,/\,(i\hbar) &= \left(2f'hs^2 + 3fh - 2gh's^2 - gh\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

We get now $[r_1, r_2]$,

$$
\begin{aligned}
[r_1, r_2] / (i\hbar) &= [r_0, r_2] / (i\hbar) + [r', r_2] / (i\hbar) \\
&= -hs^2 (\underline{x} \times \underline{k}) \cdot \underline{s}\tilde{\omega} + \left(2f'hs^2 + 3fh - 2gh's^2 - gh\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega} \\
[r_1, r_2] / (i\hbar) &= -hs^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2f'hs^2 + 3fh - 2gh's^2 - gh + h\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
\end{aligned}
$$

r_2^2,

$$
\begin{aligned}
2r_2^2 / (i\hbar) &= \left[hs^2 (\underline{z} \times \underline{x}) \cdot \underline{\theta}, hs^2 (\underline{z} \times \underline{x}) \cdot \underline{\theta}\right] / (i\hbar) \\
&= 2hs^2 / (i\hbar) \left[(\underline{z} \times \underline{x}) \cdot \underline{\theta}, hs^2\right] (\underline{z} \times \underline{x}) \cdot \underline{\theta} \\
&= -2hs^2 / (i\hbar) \left[\underline{k} \cdot \underline{\theta}, hs^2\right] (\underline{z} \times \underline{x}) \cdot \underline{\theta} \\
&= 2hs^2 \left(2h's^2 + 2h\right) \underline{s} \cdot \underline{\theta} (\underline{z} \times \underline{x}) \cdot \underline{\theta} \\
&= -2h \left(2h's^2 + 2h\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
\end{aligned}
$$

$$
r_2^2 = -\left(2hh's^2 + 2h^2\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
$$

And we get now $r^2 / (i\hbar) = \left(r_1^2 + [r_1, r_2] + r_2^2\right) / (i\hbar)$,

$$
r_1^2 / (i\hbar) = \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2gf's^2 + gf - f^2 - \frac{2f}{3} + \frac{g}{3} - \frac{1}{9}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
$$

$$
\begin{aligned}
r^2 / (i\hbar) &= \left(r_1^2 + [r_1, r_2] + r_2^2\right) / (i\hbar) \\
&= \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2gf's^2 + gf - f^2 - \frac{2f}{3} + \frac{g}{3} - \frac{1}{9}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega} \\
&\quad -hs^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2f'hs^2 + 3fh - 2gh's^2 - gh + h\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega} \\
&\quad - \left(2hh's^2 + 2h^2\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega} \\
r^2 / (i\hbar) &= \left(\frac{1}{9} + \frac{f}{3} - \frac{2g}{3} - h\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \begin{pmatrix} 2gf's^2 + gf - f^2 - \frac{2f}{3} + \frac{g}{3} - \frac{1}{9} \\ +2f'hs^2 + 3fh - 2gh's^2 - gh + h \\ -2hh's^2 - 2h^2 \end{pmatrix} s^2 \underline{z} \cdot \underline{s}\tilde{\omega}
\end{aligned}
$$

Now we put it all into $0 = \Omega - Dr + \hat{d}r + r^2/(i\hbar)$,

$$\begin{aligned}
0 &= \left(-\frac{1}{9}+\frac{2g}{3}-\frac{f}{3}+h\right)s^2\underline{p}\cdot\underline{s}\tilde{\omega}+\left(\left(2f'-2h'\right)s^2+3f+g-2h\right)\underline{z}\cdot\underline{s}\tilde{\omega} \\
&\quad +\left(\frac{1}{9}+\frac{f}{3}-\frac{2g}{3}-h\right)s^2\underline{p}\cdot\underline{s}\tilde{\omega}+\left(\begin{array}{c} 2gf's^2+gf-f^2-\frac{2f}{3}+\frac{g}{3}-\frac{1}{9} \\ +2f'hs^2+3fh-2gh's^2-gh+h \\ -2hh's^2-2h^2 \end{array}\right)s^2\underline{z}\cdot\underline{s}\tilde{\omega} \\
&= \left(\left(\begin{array}{c} 2gf's^2+gf-f^2-\frac{2f}{3}+\frac{g}{3}-\frac{1}{9} \\ +2f'hs^2+3fh-2gh's^2-gh+h \\ -2hh's^2-2h^2+(2f'-2h') \end{array}\right)s^2+3f+g-2h\right)\underline{z}\cdot\underline{s}\tilde{\omega}
\end{aligned}$$

$$\left(\left(\begin{array}{c} 2gf's^2+gf-f^2-\frac{2f}{3}+\frac{g}{3}-\frac{1}{9} \\ +2f'hs^2+3fh-2gh's^2-gh+h \\ -2hh's^2-2h^2+(2f'-2h') \end{array}\right)s^2+3f+g-2h\right)=0$$

$$\left(2gf's^2+gf+\frac{g}{3}-2gh's^2-gh\right)s^2+g=+2h-3f+\left(\begin{array}{c} \left(f+\frac{1}{3}\right)^2-2f'hs^2-3fh-h \\ +2hh's^2+2h^2-2f'+2h' \end{array}\right)s^2$$

solving for g we get:

$$g=\frac{\left(\begin{array}{c} \left(f+\frac{1}{3}\right)^2-2f'hs^2-3fh-h \\ +2hh's^2+2h^2-2f'+2h' \end{array}\right)s^2+2h-3f}{\left(f+\frac{1}{3}+2f's^2-2h's^2-h\right)s^2+1} \tag{B.20}$$

which is (4.12).

$h=0$ simplifies this expression somewhat:

$$g=\frac{\left(\left(f+\frac{1}{3}\right)^2-2f'\right)s^2-3f}{\left(\left(f+\frac{1}{3}\right)+2f's^2\right)s^2+1} \tag{B.21}$$

<u>Constant Solution:</u>

To simplify things more we find the solution to the above that has f, g, and h as constants. We substitute, in (B.20) $f'=g'=h'=0$.

$$g=\frac{\left(\left(f+\frac{1}{3}\right)^2-3fh-h+2h^2\right)s^2+2h-3f}{\left(f-h+\frac{1}{3}\right)s^2+1}$$

we express the above as a polynomial in s^2 and set the coefficients to zero:

$$0 = \left(\left(f + \frac{1}{3}\right)^2 - 3fh - h + 2h^2 - g\left(f - h + \frac{1}{3}\right)\right) s^2 + 2h - 3f - g$$

so we need:

$$g = 2h - 3f$$

and:

$$\left(f + \frac{1}{3}\right)^2 - 3fh - h + 2h^2 - g\left(f - h + \frac{1}{3}\right) = 0$$

by substituting the first into the second we get:

$$\left(f + \frac{1}{3}\right)^2 - 3fh - h + 2h^2 - (2h - 3f)\left(f - h + \frac{1}{3}\right) = 0$$

Expanding and factoring we get:

$$\frac{1}{9}\left(3\left(f - h\right) + 1\right)\left(12\left(f - h\right) + 1\right) = 0$$

so $\{f - h = -1/3, g = 2h - 3f\}$ and $\{f - h = -1/12, g = 2h - 3f\}$ are both solutions. We can simplify these further to two sets of constant solutions:

$$\{f = -1/3 + h, g = 1 - h\}$$

and:

$$\{f = -1/12 + h, g = 1/4 - h\}$$

To simplify things a bit, we choose the solution when $\{h = 0, f = -1/3, g = 1\}$. In this case:

$$r = -\frac{1}{3}\left(\underline{p} \cdot \underline{s}\right)\left(\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta}\right) + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{s} \cdot \underline{\theta} \tag{B.22}$$

which is equation (4.13).

QED.

B.3 DERIVATION OF IDENTITIES (B.14) THROUGH (B.18)

This Appendix is a derivation of identities (B.14) through (B.18) (except (B.15)) in Appendix B.1.

*Note that identities (B.2) through (B.8) are used very frequently in these calculations (especially (B.7) and (B.8)).

Proof of (B.14):

$$\tilde{D}\underline{s} = D\underline{s} - \hat{d}\underline{s} - [r, \underline{s}] / (i\hbar) = \underline{\theta} \times \underline{s} - \underline{\theta} - [r, \underline{s}] / (i\hbar)$$

knowing (4.7) and:

$$\begin{aligned} [r, \underline{s}] / (i\hbar) &= \left[-\frac{1}{3} \left(\underline{p} \cdot \underline{s} \right) \left(\left(\underline{x} \times \underline{s} \right) \cdot \underline{\theta} \right) + \left(\underline{z} \times \underline{x} \right) \cdot \underline{ss} \cdot \underline{\theta}, \underline{s} \right] / (i\hbar) \\ &= [-k_\mu, \underline{s}] \, s^\mu \left(\underline{s} \cdot \underline{\theta} \right) / (i\hbar) = \underline{s} \left(\underline{s} \cdot \underline{\theta} \right) \end{aligned}$$

using (B.2) and (B.3).

to get:

$$\tilde{D}\underline{s} = \underline{\theta} \times \underline{s} - \underline{\theta} - \underline{s} \left(\underline{s} \cdot \underline{\theta} \right)$$

which is (B.14).

QED.

Proof of (B.18) is easily seen using (B.14):

$$\tilde{D}s^2 = 2\underline{s} \cdot \tilde{D}\underline{s} = -2 \left(s^2 + 1 \right) \left(\underline{s} \cdot \underline{\theta} \right)$$

which is (B.18).

QED.

Proof of (B.16):

In a similar manner we get $\tilde{D}\underline{k}$:

$$\begin{aligned} \tilde{D}\underline{k} &= D\underline{k} - \hat{d}\underline{k} - [r, \underline{k}] / (i\hbar) \\ &= \underline{\theta} \times \underline{k} - \frac{2}{3} \underline{\theta} \times \underline{x} \left(\underline{p} \cdot \underline{s} \right) + \frac{1}{3} \left(\underline{p} \cdot \underline{\theta} \right) \left(\underline{s} \times \underline{x} \right) - \underline{\alpha} - [r, \underline{k}] / (i\hbar) \end{aligned}$$

$$
\begin{aligned}
\left[r,\underline{k}\right]/\left(i\hbar\right) &= -\frac{1}{3}\left[\left(\underline{p}\cdot\underline{s}\right),\underline{k}\right]\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)/\left(i\hbar\right)-\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\left[\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right),\underline{k}\right]/\left(i\hbar\right) \\
&\quad +z_{\mu}\left[\left(\underline{x}\times\underline{s}\right)^{\mu}\left(\underline{s}\cdot\underline{\theta}\right),\underline{k}\right]/\left(i\hbar\right) \\
&= -\frac{1}{3}\underline{p}\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)-\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\left(\varepsilon_{\mu\nu\rho}x^{\mu}\theta^{\rho}\left[s^{\nu},\underline{k}\right]\right)/\left(i\hbar\right) \\
&\quad +z_{\mu}\left(\varepsilon^{\mu\nu\rho}x_{\nu}\left[s_{\rho},\underline{k}\right]\left(\underline{s}\cdot\underline{\theta}\right)+\left(\underline{x}\times\underline{s}\right)^{\mu}\left[\left(\underline{s}\cdot\underline{\theta}\right),\underline{k}\right]\right)/\left(i\hbar\right) \\
&= -\frac{1}{3}\underline{p}\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)+\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\left(\underline{x}\times\underline{\theta}\right)+\underline{z}\times\underline{x}\left(\underline{s}\cdot\underline{\theta}\right) \\
&\quad +\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{\theta}
\end{aligned}
$$

knowing (4.7).

By expanding some terms in the $\{\underline{x},\underline{s},\underline{x}\times\underline{s}\}$ basis using (B.10) and (B.11) we find:

$$
\begin{aligned}
\tilde{D}\underline{k} &= \underline{\theta}\times\underline{k}-\frac{2}{3}\left(\underline{\theta}\times\underline{x}\right)\left(\underline{p}\cdot\underline{s}\right)-\underline{\alpha} \\
&\quad +\frac{1}{3}\left(\underline{p}\cdot\underline{\theta}\right)\left(\underline{s}\times\underline{x}\right)+\frac{1}{3}\underline{p}\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right) \\
&\quad -\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\left(\underline{x}\times\underline{\theta}\right)-\underline{z}\times\underline{x}\left(\underline{s}\cdot\underline{\theta}\right)-\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{\theta} \\
&= \underline{\theta}\times\underline{k}+\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\underline{x}\times\underline{\theta}-\underline{\alpha} \\
&\quad -\frac{1}{3}\left(\underline{\theta}\cdot\underline{s}\right)\left(\underline{p}\cdot\underline{s}\right)\frac{1}{s^{2}}\underline{x}\times\underline{s}+\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\frac{1}{s^{2}}\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)\underline{s} \\
&\quad -\underline{z}\times\underline{x}\left(\underline{s}\cdot\underline{\theta}\right)-\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{\theta} \\
&= \underline{\theta}\times\underline{k}+\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\underline{x}\times\underline{\theta}-\underline{\alpha} \\
&\quad +\frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\frac{1}{s^{2}}\left(-\underline{s}\cdot\left(\underline{x}\times\underline{\theta}\right)\underline{s}-\left(\left(\underline{x}\times\underline{\theta}\right)\cdot\left(\underline{x}\times\underline{s}\right)\right)\underline{x}\times\underline{s}\right) \\
&\quad -\underline{z}\times\underline{x}\left(\underline{s}\cdot\underline{\theta}\right)-\underline{z}\cdot\left(\underline{x}\times\underline{s}\right) \\
\tilde{D}\underline{k} &= \underline{\theta}\times\underline{k}-\underline{\alpha}-\underline{z}\times\underline{x}\left(\underline{s}\cdot\underline{\theta}\right)-\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{\theta}
\end{aligned}
$$

$$
\tilde{D}\underline{k}=\underline{\theta}\times\underline{k}-\underline{\alpha}-\underline{z}\times\underline{x}\left(\underline{s}\cdot\underline{\theta}\right)-\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{\theta}
$$

which is (B.16).

QED.

Proof of (B.17)**:**

$\boxed{\tilde{D}\underline{z}}$

In a similar manner we get $\tilde{D}\underline{z}$:

$$\begin{aligned}
\tilde{D}\underline{z} &= \tilde{D}\underline{p} - \left(\tilde{D}\underline{x}\right) \times \underline{k} - \underline{x} \times \left(\tilde{D}\underline{k}\right) \\
&= d\underline{p} - (\underline{\theta} \times \underline{x}) \times \underline{k} - \underline{x} \times \left(\tilde{D}\underline{k}\right) \\
&= \underline{\alpha} \times \underline{x} - \underline{p} \times \underline{\theta} - (\underline{k} \cdot \underline{\theta})\, \underline{x} - \underline{x} \times \left(\tilde{D}\underline{k}\right) \\
&= -\underline{p} \times \underline{\theta} - (\underline{k} \cdot \underline{\theta})\, \underline{x} + \underline{z}\, (\underline{s} \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{x} \times \underline{\theta} \\
&= -\left((\underline{x} \times \underline{z}) \cdot \underline{\theta}\right) \underline{x} + \underline{z}\, (\underline{s} \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{x} \times \underline{\theta} \\
\tilde{D}\underline{z} &= \underline{\theta} \times \underline{z} + \underline{z}\, (\underline{s} \cdot \underline{\theta}) + \underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{x} \times \underline{\theta}
\end{aligned}$$

which is (B.17).

QED.

B.4 DERIVATION OF THE SOLUTIONS (4.12) AND (4.22)

In this Appendix we start with the ansatz below for the solution of (6.17) and (6.18):

$$\hat{\underline{x}} = v\left(s^2\right) \underline{x} + w\left(s^2\right) \underline{x} \times \underline{s} + y\left(s^2\right) \underline{s}$$

$$\hat{\underline{p}} = \left(\underline{z} \cdot \underline{s}t\left(s^2\right) + \underline{z} \cdot (\underline{x} \times \underline{s})\, q\left(s^2\right)\right) \underline{x} + \underline{z}n\left(s^2\right) + \underline{z} \times \underline{x}u\left(s^2\right)$$

We show that a solution is (4.21) and (4.22) i.e.:

$$\hat{\underline{x}} = (\underline{x} - \underline{x} \times \underline{s})\, \frac{1}{\sqrt{s^2+1}}$$

$$\hat{\underline{p}} = (\underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{x} + \underline{z})\, \sqrt{s^2+1}$$

*Note that identities (B.2) through (B.8) are used very frequently in these calculations (especially (B.7) and (B.8)). Of course when we calculate the $\tilde{D}$ of something we will use the identities (B.14) through (B.18).

B.4.1 Proof of the Solution in (4.21)

The condition we need to compute is (6.17):

$$\left(D-\hat{D}\right)\hat{x}^{\mu}=0 \quad , \quad \sigma\left(\hat{x}^{\mu}\right)=b^{\mu}_{0,0}=x^{\mu}$$

Proof of (4.21):
Remember that $\tilde{D}=\left(D-\hat{D}\right)$:

$$\begin{aligned}\tilde{D}\underline{\hat{x}} &= -2v'\left(s^2+1\right)\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}+v\left(\underline{\theta}\times\underline{x}\right)-2w'\left(s^2+1\right)\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}\times\underline{s}\\ &+w\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}-w\underline{x}\times\underline{\theta}-w\underline{x}\times\underline{s}\left(\underline{s}\cdot\underline{\theta}\right)-2y'\left(s^2+1\right)\left(\underline{s}\cdot\underline{\theta}\right)\underline{s}\\ &+y\left(\underline{\theta}\times\underline{s}-\underline{\theta}-\underline{s}\left(\underline{s}\cdot\underline{\theta}\right)\right)\end{aligned}$$

Expanding the vectors in the $\{\underline{x},\underline{s},\underline{x}\times\underline{s}\}$ basis using (B.10) and (B.11) and collecting we find:

$$\begin{aligned}\tilde{D}\underline{\hat{x}} &= \left(-2v'\left(s^2+1\right)\left(\underline{s}\cdot\underline{\theta}\right)+w\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}\right)\underline{x}\\ &+\left(-2y'\left(s^2+1\right)\left(\underline{s}\cdot\underline{\theta}\right)-y\left(\underline{s}\cdot\underline{\theta}\right)\right)\underline{s}\\ &+\left(-2w'\left(s^2+1\right)\left(\underline{s}\cdot\underline{\theta}\right)-w\left(\underline{s}\cdot\underline{\theta}\right)\right)\underline{x}\times\underline{s}\\ &+(v+w)\left(\left(\left(\underline{\theta}\times\underline{x}\right)\cdot\left(\underline{x}\times\underline{s}\right)\underline{x}\times\underline{s}+\left(\left(\underline{\theta}\times\underline{x}\right)\cdot\underline{s}\right)\underline{s}\right)\frac{1}{s^2}\\ &+y\left(\left(\underline{\theta}\times\underline{s}\right)\cdot\left(\underline{x}\times\underline{s}\right)\underline{x}\times\underline{s}+\left(\left(\underline{\theta}\times\underline{s}\right)\cdot\underline{s}\right)\underline{s}\right)\frac{1}{s^2}\\ &-y\left(\underline{\theta}\cdot\left(\underline{x}\times\underline{s}\right)\underline{x}\times\underline{s}+\left(\underline{\theta}\cdot\underline{s}\right)\underline{s}\right)\frac{1}{s^2}\end{aligned}$$

$$\begin{aligned}\tilde{D}\underline{\hat{x}} &= \left(-2v'\left(s^2+1\right)+w\right)\left(\underline{s}\cdot\underline{\theta}\right)\underline{x}\\ &+\begin{pmatrix}\left(-2y'\left(s^2+1\right)s^2-ys^2-y\right)\left(\underline{s}\cdot\underline{\theta}\right)\\ +(v+w)\left(\left(\underline{\theta}\times\underline{x}\right)\cdot\underline{s}\right)\end{pmatrix}\frac{1}{s^2}\underline{s}\\ &+\begin{pmatrix}\left(-2w'\left(s^2+1\right)s^2-ws^2-(v+w)\right)\left(\underline{s}\cdot\underline{\theta}\right)\\ -y\underline{\theta}\cdot\left(\underline{x}\times\underline{s}\right)\end{pmatrix}\frac{1}{s^2}\underline{x}\times\underline{s}\end{aligned}$$

Knowing that from (B.9) we have:

$$\underline{x} \cdot \left(\tilde{D}\hat{\underline{x}}\right) (\underline{x} \times \underline{s}) \cdot \underline{\theta} = \left(-2v' \left(s^2 + 1\right) + w\right) \tilde{\omega}$$

now it is well-known that $\mathbb{S}^2$ is a symplectic manifold in its own right with symplectic form $\tilde{\omega} = \underline{x} \cdot (\underline{\theta} \times \underline{\theta}) /2$. This means explicitly that $\tilde{\omega}_{\mu\nu}$ is invertible, i.e., there exists an $\left(\tilde{\omega}^{-1}\right)^{\mu\nu}$ such that $\left(\tilde{\omega}^{-1}\right)_{\mu}^{\ \nu} \tilde{\omega}_{\nu\rho} = g_{\mu\rho}$ where $g_{\mu\rho}$ is the sphere metric in formula (4.2). This is easily observed by by knowing $\tilde{\omega}_{\mu\nu} = \varepsilon_{\mu\nu\rho} x^{\rho}/2$ thus $\left(\tilde{\omega}^{-1}\right)^{\mu\nu} = -2\varepsilon^{\mu\nu\rho} x_{\rho}$. The invertibility of this two-form guarantees us that any equation of the form $b\tilde{\omega} = 0$ where b is a zero form is satisfied iff $b = 0$. We also have the obvious identities $\left((\underline{\theta} \times \underline{x}) \cdot \underline{s}\right)^2 = (\underline{s} \cdot \underline{\theta})^2 = 0$ (because the square of any form is zero by the antisymmetry of the wedge product). Using the orthogonality of $\{\underline{x}, \underline{s}, \underline{x} \times \underline{s}\}$ and these formulas we can pick out the terms we need. For example:

$$\underline{s} \cdot \left(\tilde{D}\hat{\underline{x}}\right) \left((\underline{\theta} \times \underline{x}) \cdot \underline{s}\right) = \left(-2y' \left(s^2 + 1\right) s^2 - ys^2 - y\right) \tilde{\omega}$$

We utilize this strategy to solve the equation above for v, w, and y. Therefore:

$$0 = \underline{x} \cdot \left(\tilde{D}\hat{\underline{x}}\right) (\underline{x} \times \underline{s}) \cdot \underline{\theta} = \left(-2v' \left(s^2 + 1\right) + w\right) \tilde{\omega}$$

$$\implies \left(-2v' \left(s^2 + 1\right) + w\right) = 0$$

$$\underline{s} \cdot \left(\tilde{D}\hat{\underline{x}}\right) \left((\underline{\theta} \times \underline{x}) \cdot \underline{s}\right) = \left(-2y' \left(s^2 + 1\right) s^2 - ys^2 - y\right) \tilde{\omega}$$

$$\implies \left(-2y' \left(s^2 + 1\right) s^2 - ys^2 - y\right) = 0$$

$$\underline{s} \cdot \left(\tilde{D}\hat{\underline{x}}\right) (\underline{s} \cdot \underline{\theta}) = -\left(v + w\right) \tilde{\omega}$$

$$\implies v = w$$

$$(\underline{x} \times \underline{s}) \cdot \left(\tilde{D}\hat{\underline{x}}\right) \left((\underline{\theta} \times \underline{x}) \cdot \underline{s}\right) = \left(-2w' \left(s^2 + 1\right) s^2 - ws^2 - (v + w)\right) \tilde{\omega}$$

$$\implies \left(-2w' \left(s^2 + 1\right) s^2 - ws^2 - (v + w)\right) = 0$$

$$(\underline{x} \times \underline{s}) \cdot \left(\tilde{D}\hat{\underline{x}}\right) (\underline{s} \cdot \underline{\theta}) = y\tilde{\omega}$$

$$\implies y = 0$$

All we need to do is solve the resulting five equations above:

$$\left(-2v' \left(s^2 + 1\right) + w\right) = 0 \tag{B.23}$$

$$\left(-2y'\left(s^2+1\right)s^2-ys^2-y\right)=0 \tag{B.24}$$

$$\left(-2w'\left(s^2+1\right)s^2-ws^2-(v+w)\right)=0 \tag{B.25}$$

$$v=-w \tag{B.26}$$

$$y=0 \tag{B.27}$$

These reduce to solving the one equation knowing $v=-w$ and $y=0$:

$$-2v'\left(s^2+1\right)-v=0 \tag{B.28}$$

which has solution:

$$v=\frac{A}{\sqrt{s^2+1}}$$

and:

$$\underline{\hat{x}}=\left(\underline{x}-\underline{x}\times\underline{s}\right)\frac{A}{\sqrt{s^2+1}}$$

where A is some constant which we determine by the condition $\sigma\left(\underline{\hat{x}}\right)=\underline{x}$ in (6.17):

$$\sigma\left(\underline{\hat{x}}\right)=A\underline{x}\implies A=1$$

and so:

$$\underline{\hat{x}}=\left(\underline{x}-\underline{x}\times\underline{s}\right)\frac{1}{\sqrt{s^2+1}} \tag{B.29}$$

which is the solution in (4.21).

QED.

B.4.2 Proof of the Solution in (4.22)

The condition we need to compute is (6.18):

$$\left(D-\hat{D}\right)\hat{p}_\mu = 0 \quad , \quad \sigma\left(\hat{p}_\mu\right) = c_{0,0,\mu} = p_\mu$$

Proof of (4.22):

The strategy to solve for the functions t, q, n, and u is identical to the procedure for $\underline{\hat{x}}$. The calculations are further complicated by the presence of the additional vector $\underline{z}$ and the connection $\tilde{D}$ which will now introduce some $\underline{\alpha}$'s. By carefully choosing our ansatz we can cancel the $\underline{\alpha}$'s from the beginning. However, we are stuck with the $\underline{z}$ terms, but can minimize their presence by knowing that the connection $\tilde{D}$ acting on $\underline{z}$ does not introduce any terms more than linear in $\underline{z}$ (i.e., no quadratics, cubics, etc.). We can thus simplify our ansatz to be:

$$\underline{\hat{p}} = \left(\underline{z}\cdot\underline{s}t\left(s^2\right) + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)q\left(s^2\right)\right)\underline{x} + \underline{z}n\left(s^2\right) + \underline{z}\times\underline{x}u\left(s^2\right)$$

The procedure is the same as in the case for $\underline{\hat{x}}$ above. We will compute $\tilde{D}\underline{\hat{p}} = 0$ and use the invertibility of $\tilde{\omega}$ to obtain differential equations for the unknown functions t, q, n, and u.

$$\begin{aligned}\tilde{D}\underline{\hat{p}} \;=\; & \left(\begin{array}{c}\left(\tilde{D}\underline{z}\right)\cdot\underline{s}t + \underline{z}\cdot\left(\tilde{D}\underline{s}\right)t + \underline{z}\cdot\underline{s}\left(\tilde{D}t\right)\\ +\left(\tilde{D}\underline{z}\right)\cdot\left(\underline{x}\times\underline{s}\right)q + \underline{z}\cdot\left(\left(\tilde{D}\underline{x}\right)\times\underline{s}\right)q + \underline{z}\cdot\left(\underline{x}\times\left(\tilde{D}\underline{s}\right)\right)q\\ +\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\tilde{D}q\end{array}\right)\underline{x}\\ & +\left(\underline{z}\cdot\underline{s}t + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)q\right)\tilde{D}\underline{x}\\ & +\left(\tilde{D}\underline{z}\right)n + \underline{z}\left(\tilde{D}n\right) + \left(\tilde{D}\underline{z}\right)\times\underline{x}u + \underline{z}\times\left(\tilde{D}\underline{x}\right)u + \underline{z}\times\underline{x}\left(\tilde{D}u\right)\end{aligned}$$

$$\underline{z}\times\left(\tilde{D}\underline{x}\right)u = \underline{z}\times\left(\underline{\theta}\times\underline{x}\right)u = -\left(\underline{z}\cdot\underline{\theta}\right)u\underline{x}$$

$$= \left(\begin{array}{c} \left(\tilde{D}\underline{z}\right) \cdot \left(\underline{s}t + \left(\underline{x} \times \underline{s}\right) q\right) \\ +\underline{z} \cdot \left(\tilde{D}\underline{s}\right) t - 2\underline{z} \cdot \underline{s} \left(\tilde{D}t\right) t' \left(s^2+1\right) \left(\underline{s} \cdot \underline{\theta}\right) \\ +\underline{z} \cdot \left(\underline{x} \times \left(\tilde{D}\underline{s}\right)\right) q \\ -2\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q' \left(s^2+1\right) \left(\underline{s} \cdot \underline{\theta}\right) - \left(\underline{z} \cdot \underline{\theta}\right) u\underline{x} \end{array} \right) \underline{x}$$

$$\begin{aligned} &+ \left(\underline{z} \cdot \underline{s}t + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q\right) \left(\underline{\theta} \times \underline{x}\right) - 2\underline{z}n' \left(s^2+1\right) \left(\underline{s} \cdot \underline{\theta}\right) \\ &+ \left(\tilde{D}\underline{z}\right) n \\ &+ \left(\tilde{D}\underline{z}\right) \times \underline{x}u \\ &-2 \left(\underline{z} \times \underline{x}\right) u' \left(s^2+1\right) \left(\underline{s} \cdot \underline{\theta}\right) \end{aligned}$$

$$= \left(\begin{array}{c} + \left(\begin{array}{c} \underline{z} \cdot \underline{s}t + 2\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q - \underline{z} \cdot \underline{s}t \\ -2\underline{z} \cdot \underline{s}t' \left(s^2+1\right) - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q \\ -2\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q' \left(s^2+1\right) \end{array} \right) \left(\underline{s} \cdot \underline{\theta}\right) \\ + \left(-\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) t\right) \underline{\theta} \cdot \left(\underline{x} \times \underline{s}\right) \\ -\underline{z} \cdot \left(\underline{x} \times \underline{\theta}\right) q + \left(\underline{z} \cdot \underline{\theta}\right) \left(u - t\right) \end{array} \right) \underline{x}$$

$$\begin{aligned} &+ \left(\underline{z} \cdot \underline{s}t + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q\right) \left(\underline{\theta} \times \underline{x}\right) - 2\underline{z}n' \left(s^2+1\right) \left(\underline{s} \cdot \underline{\theta}\right) \\ &+ \left(\underline{\theta} \times \underline{z} + \underline{z} \left(\underline{s} \cdot \underline{\theta}\right) + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{x} \times \underline{\theta}\right) n \\ &+ \left(\underline{z} \times \underline{x} \left(\underline{s} \cdot \underline{\theta}\right) + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{\theta}\right) u \\ &-2 \left(\underline{z} \times \underline{x}\right) u' \left(s^2+1\right) \left(\underline{s} \cdot \underline{\theta}\right) \end{aligned}$$

$$= \left(\begin{array}{c} + \left(\begin{array}{c} \underline{z} \cdot \underline{s}t + 2\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q - \underline{z} \cdot \underline{s}t \\ -2\underline{z} \cdot \underline{s}t' \left(s^2+1\right) - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q \\ -2\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q' \left(s^2+1\right) \end{array} \right) \left(\underline{s} \cdot \underline{\theta}\right) \\ + \left(-\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) t\right) \underline{\theta} \cdot \left(\underline{x} \times \underline{s}\right) \\ + \left(\underline{z} \times \underline{x}\right) \cdot \underline{\theta} \left(-q + n\right) + \left(\underline{z} \cdot \underline{\theta}\right) \left(-u - t\right) \end{array} \right) \underline{x}$$

$$\begin{aligned} &+\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) u\underline{\theta} \\ &+ \left(\underline{z} \cdot \underline{s}t + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) n\right) \left(\underline{\theta} \times \underline{x}\right) \\ &+\underline{z} \left(n - 2n' \left(s^2+1\right)\right) \left(\underline{s} \cdot \underline{\theta}\right) \\ &+\underline{z} \times \underline{x} \left(u - 2u' \left(s^2+1\right)\right) \left(\underline{s} \cdot \underline{\theta}\right) \end{aligned}$$

Expanding the vectors in the $\{\underline{x}, \underline{s}, \underline{x} \times \underline{s}\}$ basis using (B.10) and (B.11) (including all $\underline{\theta}$'s) and collecting we find:

$$
= \left(\begin{array}{c}
+ \left(\begin{array}{c} \underline{z} \cdot \underline{s} t + 2\underline{z} \cdot (\underline{x} \times \underline{s})\, q - \underline{z} \cdot \underline{s} t \\ -2\underline{z} \cdot \underline{s} t' \left(s^2+1\right) - \underline{z} \cdot (\underline{x} \times \underline{s})\, q \\ -2\underline{z} \cdot (\underline{x} \times \underline{s})\, q' \left(s^2+1\right) \end{array} \right) (\underline{s} \cdot \underline{\theta}) \\
+ \left(-\underline{z} \cdot (\underline{x} \times \underline{s})\, t\right) \underline{\theta} \cdot (\underline{x} \times \underline{s}) \\
+ (\underline{z} \times \underline{x}) \cdot \underline{\theta} \,(-q+n) + (\underline{z} \cdot \underline{\theta})\,(-u-t)
\end{array} \right) \underline{x}
$$

$$
+ \left(\begin{array}{c}
\left(\begin{array}{c} \underline{z} \cdot (\underline{x} \times \underline{s})\, u \\ + (\underline{z} \cdot \underline{s}) \left(n - 2n' \left(s^2+1\right)\right) \\ (\underline{z} \cdot (\underline{x} \times \underline{s})) \left(u - 2u' \left(s^2+1\right)\right) \end{array} \right) (\underline{s} \cdot \underline{\theta}) \\
\left(\begin{array}{c} \underline{z} \cdot \underline{s} t + \underline{z} \cdot (\underline{x} \times \underline{s})\, q \\ -\underline{z} \cdot (\underline{x} \times \underline{s})\, n \end{array} \right) \underline{\theta} \cdot (\underline{x} \times \underline{s})
\end{array} \right) \frac{1}{s^2} \underline{s}
$$

$$
+ \left(\begin{array}{c}
\left(\underline{z} \cdot (\underline{x} \times \underline{s})\, u\right) \underline{\theta} \cdot (\underline{x} \times \underline{s}) \\
\left(\begin{array}{c} -\underline{z} \cdot \underline{s} t - \underline{z} \cdot (\underline{x} \times \underline{s})\, q \\ +\underline{z} \cdot (\underline{x} \times \underline{s})\, n \\ +\underline{z} \cdot (\underline{x} \times \underline{s}) \left(n - 2n' \left(s^2+1\right)\right) \\ - (\underline{z} \cdot \underline{s}) \left(u - 2u' \left(s^2+1\right)\right) \end{array} \right) (\underline{s} \cdot \underline{\theta})
\end{array} \right) \frac{1}{s^2} \underline{x} \times \underline{s}
$$

$$
= \left(\begin{array}{c}
+ \left(\begin{array}{c} \underline{z} \cdot \underline{s} \left(\frac{1}{s^2} (-u-t) - 2t' \left(s^2+1\right)\right) \\ \underline{z} \cdot (\underline{x} \times \underline{s}) \left(-2q' \left(s^2+1\right) + \frac{1}{s^2} (-q+n) + q\right) \end{array} \right) (\underline{s} \cdot \underline{\theta}) \\
+ \left(\begin{array}{c} \underline{z} \cdot (\underline{x} \times \underline{s}) \left(-t + \frac{1}{s^2} (-u-t)\right) \\ - (\underline{z} \cdot \underline{s}) \frac{1}{s^2} (-q+n) \end{array} \right) \underline{\theta} \cdot (\underline{x} \times \underline{s})
\end{array} \right) \underline{x}
$$

$$
+ \left(\begin{array}{c}
\left(\begin{array}{c} \underline{z} \cdot (\underline{x} \times \underline{s}) \left(2u - 2u' \left(s^2+1\right)\right) \\ + (\underline{z} \cdot \underline{s}) \left(n - 2n' \left(s^2+1\right)\right) \end{array} \right) (\underline{s} \cdot \underline{\theta}) \\
\left(\begin{array}{c} \underline{z} \cdot \underline{s} t \\ +\underline{z} \cdot (\underline{x} \times \underline{s}) (q-n) \end{array} \right) \underline{\theta} \cdot (\underline{x} \times \underline{s})
\end{array} \right) \frac{1}{s^2} \underline{s}
$$

$$
+ \left(\begin{array}{c}
\left(\underline{z} \cdot (\underline{x} \times \underline{s})\, u\right) \underline{\theta} \cdot (\underline{x} \times \underline{s}) \\
\left(\begin{array}{c} \underline{z} \cdot \underline{s} \left(-t - u + 2u' \left(s^2+1\right)\right) \\ +\underline{z} \cdot (\underline{x} \times \underline{s}) \left(-q + 2n - 2n' \left(s^2+1\right)\right) \end{array} \right) (\underline{s} \cdot \underline{\theta})
\end{array} \right) \frac{1}{s^2} \underline{x} \times \underline{s}
$$

In the same method as in the calculation for $\underline{\hat{x}}$ the equation $\tilde{D}\underline{\hat{p}} = 0$ (condition (6.18)) yields 11 conditions which are reduced to the equations:

$$q - 2q'\left(s^2 + 1\right) = 0$$

$$t = 0$$

$$q = n$$

$$u = 0$$

which yields the solution:

$$q = A\sqrt{1 + s^2}$$

and so:

$$\underline{\hat{p}} = \left(\underline{z} \cdot \left(\underline{x} \times \underline{s}\right)\underline{x} + \underline{z}\right) A\sqrt{1 + s^2}$$

where A is some constant which we determine by the condition $\sigma\left(\underline{\hat{p}}\right) = \underline{p}$ in (6.18):

$$\sigma\left(\underline{\hat{p}}\right) = A \implies A = 1$$

and finally:

$$\underline{\hat{p}} = \left(\underline{z} \cdot \left(\underline{x} \times \underline{s}\right)\underline{x} + \underline{z}\right) \sqrt{1 + s^2}$$

which is the solution in (4.22).

QED.

APPENDIX C

CONSTANT CURVATURE MANIFOLDS OF CODIMENSION ONE

C.1 PROOF THAT (5.14) YIELDS THE SOLUTION IN (6.16)

We want to show that the ansatz given in (5.14) (and (D.12)) subject to the condition in (5.15) (and (D.13)):

$$\begin{aligned}\hat{Q} &= (s^\mu \alpha_\mu - z_\mu dx^\mu) + j^\mu \alpha_\mu + z_\nu f^\nu{}_\mu dx^\mu \\ &\quad + p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - dx^\sigma \hat{\partial}_\sigma \right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma \right)\end{aligned}$$

yield the solution stated in (6.16):

$$\hat{Q} = (s^\mu \alpha_\mu - z_\mu dx^\mu) - C\,(z_\nu s^\nu)\,(s_\mu dx^\mu) + \frac{C}{3}\left((p_\nu s^\nu)\,(s_\mu dx^\mu) - (p_\nu dx^\nu)\,u\right) \tag{C.1}$$

Proof:

We want to find a solution to the equation (5.15) (and (D.13)):

$$\left(\left(D + f^\mu{}_\rho dx^\rho \hat{\partial}_\mu - dx^\mu \hat{\partial}_\mu \right) f^\nu{}_\sigma + \Gamma^\nu{}_{\rho\mu} dx^\mu f^\rho{}_\sigma - \Gamma^\nu{}_{\sigma\mu} dx^\mu + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta \right) dx^\sigma = 0$$

Let $u := \eta_{\mu\nu} s^\mu s^\nu = g_{\mu\nu} s^\mu s^\nu$ then $f^\nu{}_\mu = f(x)\, s^\nu s_\mu$:

$$\begin{aligned}
0 &= \begin{pmatrix} \left(D + f(s_\rho dx^\rho) s^\mu \hat{\partial}_\mu - dx^\mu \hat{\partial}_\mu\right)(f s^\nu s_\sigma) \\ +\Gamma^\nu{}_{\rho\mu} s^\rho dx^\mu f s_\sigma - \Gamma^\nu{}_{\sigma\mu} dx^\mu + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta \end{pmatrix} dx^\sigma \\
&= \begin{pmatrix} dx^\mu (\partial_\mu f) s^\nu s_\sigma - f\Gamma^\nu{}_{\rho\mu} dx^\mu s^\rho s_\sigma - f s^\nu \Gamma_{\sigma\rho\mu} dx^\mu s^\rho \\ +f^2 (s_\rho dx^\rho) s^\nu s_\sigma + f^2 (s_\rho dx^\rho) s_\sigma s^\nu \\ -f dx^\nu s_\sigma + f\Gamma^\nu{}_{\rho\kappa} s^\rho dx^\kappa s_\sigma - \Gamma^\nu{}_{\sigma\kappa} dx^\kappa + R^\nu{}_{\kappa\beta\sigma} s^\kappa dx^\beta \end{pmatrix} dx^\sigma \\
&= \left(\left((\partial_\mu f) s^\nu + 2f^2 s_\mu s^\nu - f\delta^\nu_\mu\right) s_\sigma - f s^\nu \Gamma_{\sigma\rho\mu} s^\rho + R^\nu{}_{\kappa\mu\sigma} s^\kappa\right) dx^\mu dx^\sigma \\
&= \left(\left((\partial_\mu f) s^\nu + 2f^2 s_\mu s^\nu - f\delta^\nu_\mu\right) s_\sigma + C\left(\delta^\nu_{[\mu} - C x_{[\mu} x^\nu\right) g_{\sigma]\kappa} s^\kappa\right) dx^\mu dx^\sigma \\
&= \left((\partial_\mu f) dx^\mu s^\nu + (C - f) dx^\nu\right) s_\sigma dx^\sigma
\end{aligned}$$

We observe that $f = C$ is a solution to (5.15) (and (D.13)). Also let $j^\mu = 0$ and putting these back into (5.14) (and (D.12)):

$$\begin{aligned}
\hat{Q} &= (s^\mu \alpha_\mu - z_\mu dx^\mu) - C(z_\nu s^\nu)(s_\mu dx^\mu) - \frac{2}{3} p_\nu R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma \\
&= (s^\mu \alpha_\mu - z_\mu dx^\mu) - C(z_\nu s^\nu)(s_\mu dx^\mu) \\
&\quad + \frac{C}{6} p_\nu \left(\left(\delta^\nu_\beta - C x_\beta x^\nu\right) g_{\sigma\mu} - 2\left(\delta^\nu_\sigma - C x_\sigma x^\nu\right) g_{\beta\mu} + \left(\delta^\nu_\mu - C x_\mu x^\nu\right) g_{\sigma\beta}\right) s^\beta s^\mu dx^\sigma \\
\hat{Q} &= (s^\mu \alpha_\mu - z_\mu dx^\mu) - C(z_\nu s^\nu)(s_\mu dx^\mu) + \frac{C}{3}\left((p_\nu dx^\nu) u - (p_\nu s^\nu)(s_\mu dx^\mu)\right)
\end{aligned}$$

$$R^\mu{}_{\nu\sigma\rho} = -C\left(\delta^\mu_{[\sigma} - C x_{[\sigma} x^\mu\right) g_{\rho]\nu}$$

$$\begin{aligned}
4R^\nu{}_{(\mu\beta)\sigma} &= -2C\left(\delta^\nu_{[\beta} - C x_{[\beta} x^\nu\right) g_{\sigma]\mu} - 2C\left(\delta^\nu_{[\mu} - C x_{[\mu} x^\nu\right) g_{\sigma]\beta} \\
&= -C\left(\left(\delta^\nu_\beta - C x_\beta x^\nu\right) g_{\sigma\mu} - 2\left(\delta^\nu_\sigma - C x_\sigma x^\nu\right) g_{\beta\mu} + \left(\delta^\nu_\mu - C x_\mu x^\nu\right) g_{\sigma\beta}\right)
\end{aligned}$$

$$\hat{Q} = (s^\mu \alpha_\mu - z_\mu dx^\mu) - C(z_\nu s^\nu)(s_\mu dx^\mu) + \frac{C}{3}\left((p_\nu s^\nu)(s_\mu dx^\mu) - (p_\nu dx^\nu) u\right)$$

where $u = s_\mu s^\mu$ and $z_\mu = p_\mu + k_\mu$ and this is the same solution as in (6.16) (and (C.1)).

C.1.1 A Check: Confirming with the Sphere Case

We want to show that the solution in (6.16) (and (C.1):

$$\hat{Q} = s^{\mu}\alpha_{\mu} - z_{\mu}dx^{\mu} - Cz_{\mu}s^{\mu}\left(s_{\nu}dx^{\nu}\right) + \frac{C}{3}\left(p_{\nu}s^{\nu}s_{\rho}dx^{\rho} - p_{\rho}dx^{\rho}u\right)$$

is confirmed by the formula obtained separately for the sphere case in (4.14):

$$\hat{Q}_{\mathbb{S}^2} = \left(\underline{s}\cdot\underline{\alpha} - \underline{k}\cdot\underline{\theta}\right) - \frac{1}{3}\left(\underline{p}\cdot\underline{s}\right)\left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right) + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{s}\cdot\underline{\theta}\right) \tag{C.2}$$

derived in Appendix B.2.

Proof:

*Note that these s^{μ} and k_{μ} are different than the two-sphere $\underline{s}$ and $\underline{k}$. The relation between them is:

$$\underline{s}_{\mathbb{S}^2} = \underline{x}\times\underline{s}_{C_{3,0}} \quad , \quad \underline{s}_{C_{3,0}} = -\underline{x}\times\underline{s}_{\mathbb{S}^2}$$

$$\underline{k}_{\mathbb{S}^2} = \underline{x}\times\underline{k}_{C_{3,0}} \quad , \quad \underline{k}_{C_{3,0}} = -\underline{x}\times\underline{k}_{\mathbb{S}^2}$$

This is because:

$$\underline{\theta}_{\mathbb{S}^2} := \underline{x}\times d\underline{x} \quad , \quad d\underline{x} = -\underline{x}\times\underline{\theta}$$

$$\underline{\alpha}_{\mathbb{S}^2} := \underline{x}\times\underline{\alpha}_{C_{3,0}} \quad , \quad \underline{\alpha}_{C_{3,0}} = -\underline{x}\times\underline{\alpha}_{\mathbb{S}^2}$$

$$\underline{z} = \underline{p} - \underline{x}\times\underline{k}$$

$$C = 1$$

Then our $\hat{Q}$ in (6.16) is:

$$-\frac{1}{3}\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{s}\cdot\underline{\theta}\right)$$

$$\begin{aligned}\hat{Q} \quad = \quad & \underline{s}\cdot\underline{\alpha} + \left(\underline{p} - \underline{x}\times\underline{k}\right)\cdot\left(\underline{x}\times\underline{\theta}\right) + \left(\underline{p} - \underline{x}\times\underline{k}\right)\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{s}\cdot\underline{\theta}\right) \\ & + \frac{1}{3}\left(-\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{s}\cdot\underline{\theta}\right) + \underline{p}\cdot\left(\underline{x}\times\underline{\theta}\right)s^2\right)\end{aligned}$$

$$\begin{aligned} = \quad & \left(\underline{s}\cdot\underline{\alpha} - \underline{k}\cdot\underline{\theta}\right) + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{s}\cdot\underline{\theta}\right) - \frac{1}{3}\underline{p}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{s}\cdot\underline{\theta}\right) \\ & + \underline{p}\cdot\left(\underline{x}\times\underline{\theta}\right)\frac{s^2}{3} + \underline{p}\cdot\left(\underline{x}\times\underline{\theta}\right)\end{aligned}$$

using (B.10) we know:

$$\begin{aligned} \underline{x} \times \underline{\theta} &= \left(\left(\underline{x} \times \underline{\theta} \right) \cdot \left(\underline{x} \times \underline{s} \right) \underline{x} \times \underline{s} + \left(\left(\underline{x} \times \underline{\theta} \right) \cdot \underline{s} \right) \underline{s} \right) \frac{1}{s^2} \\ &= \left(\underline{x} \times \underline{s} \left(\underline{s} \cdot \underline{\theta} \right) - \underline{\theta} \cdot \left(\underline{x} \times \underline{s} \right) \underline{s} \right) \frac{1}{s^2} \end{aligned}$$

then:

$$\begin{aligned} &= \left(\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta} \right) + \underline{z} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{s} \cdot \underline{\theta} \right) - \frac{1}{3} \underline{p} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{s} \cdot \underline{\theta} \right) \\ &\quad + \left\{ \left(\underline{p} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{s} \cdot \underline{\theta} \right) - \underline{\theta} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{p} \cdot \underline{s} \right) \right) \right\} \frac{1}{3} + \underline{p} \cdot \left(\underline{x} \times \underline{\theta} \right) \end{aligned}$$

$$= \left(\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta} \right) + \underline{z} \cdot \left(\underline{x} \times \underline{s} \right) \left(\underline{s} \cdot \underline{\theta} \right) - \frac{1}{3} \left(\underline{p} \cdot \underline{s} \right) \underline{\theta} \cdot \left(\underline{x} \times \underline{s} \right) + \underline{p} \cdot \left(\underline{x} \times \underline{\theta} \right)$$

So:

$$\hat{Q}_{C_{3,0}} = \hat{Q}_{\mathbb{S}^2} + \underline{p} \cdot \left(\underline{x} \times \underline{\theta} \right)$$

Because $\hat{Q}_{C_{3,0}}$ is different to $\hat{Q}_{\mathbb{S}^2}$ by $\underline{p} \cdot \left(\underline{x} \times \underline{\theta} \right)$, i.e., a term that doesn't affect the graded commutator $\left[\hat{Q}, \cdot \right]$ they are then equivalent.

QED.

C.2 THE PROOF OF SOLUTIONS IN (6.19) AND (6.20)

This appendix section is a proof that the conditions (6.17) (and (C.3) below) and (6.18) (and (C.4) below):

$$\left(D - \hat{D} \right) \hat{x}^{\mu} = D\hat{x}^{\mu} - [\hat{Q}, \hat{x}^{\mu}] / \left(i\hbar \right) = 0 \quad , \quad \sigma \left(\hat{x}^{\mu} \right) = x^{\mu} \tag{C.3}$$

$$\left(D - \hat{D} \right) \hat{p}_{\mu} = D\hat{p}_{\mu} - [\hat{Q}, \hat{p}_{\mu}] / \left(i\hbar \right) = 0 \quad , \quad \sigma \left(\hat{p}_{\mu} \right) = p_{\mu} \tag{C.4}$$

yield the solution in (6.19) and (6.20):

$$\hat{x}^{\mu} = \left(x^{\mu} + s^{\mu} \right) \frac{1}{\sqrt{Cu + 1}} \tag{C.5}$$

$$\hat{p}_{\mu} = \left(-C z_{\nu} s^{\nu} x_{\mu} + z_{\mu} \right) \sqrt{Cu + 1} - iC\hbar n \hat{x}_{\mu} \tag{C.6}$$

C.2.1 Proof of (6.19)

We will start off with the ansatz:

$$\hat{x}^{\mu} = f\left(x,s\right)x^{\mu} + h\left(x,s\right)s^{\mu} \tag{C.7}$$

and require that the condition (C.3) is satisfied:

$$\left(D-\hat{D}\right)\hat{x}^{\mu} = D\hat{x}^{\mu} - [\hat{Q},\hat{x}^{\mu}]/\left(i\hbar\right) = 0 \quad , \quad \sigma\left(\hat{x}^{\mu}\right) = x^{\mu}$$

where $\hat{Q}$ is:

$$\hat{Q} = \left(s^{\mu}\alpha_{\mu} - z_{\mu}dx^{\mu}\right) - C\left(z_{\nu}s^{\nu}\right)\left(s_{\mu}dx^{\mu}\right) + \frac{C}{3}\left(\left(p_{\nu}dx^{\nu}\right)u - \left(p_{\nu}s^{\nu}\right)\left(s_{\mu}dx^{\mu}\right)\right)$$

where $u = s_{\mu}s^{\mu}$ and $z_{\mu} = p_{\mu} + k_{\mu}$ (see (6.16)).

Proof of (6.19):

$$\begin{aligned} D\hat{x}^{\mu} &= dx^{\nu}\nabla_{\nu}\left(fx^{\mu}\right) + dx^{\nu}\nabla_{\nu}\left(hs^{\mu}\right) \\ &= dx^{\nu}\left(\partial_{\nu}f - \Gamma^{\rho}{}_{\sigma\nu}s^{\sigma}\hat{\partial}_{\rho}f\right)x^{\mu} + fdx^{\mu} + dx^{\nu}\left(\partial_{\nu}h - \Gamma^{\rho}{}_{\sigma\nu}s^{\sigma}\hat{\partial}_{\rho}h\right)s^{\mu} - hdx^{\nu}\Gamma^{\mu}{}_{\sigma\nu}s^{\sigma} \end{aligned}$$

$$\begin{aligned} -[\hat{Q},\hat{x}^{\mu}]/\left(i\hbar\right) &= \left[z_{\sigma}dx^{\sigma} + C\left(z_{\nu}s^{\nu}\right)\left(s_{\sigma}dx^{\sigma}\right), fx^{\mu} + hs^{\mu}\right]/\left(i\hbar\right) \\ &= \left[z_{\sigma},f\right]x^{\mu}dx^{\sigma}/\left(i\hbar\right) + C\left[z_{\nu},f\right]s^{\nu}x^{\mu}\left(s_{\sigma}dx^{\sigma}\right)/\left(i\hbar\right) \\ &\quad + \left[z_{\sigma},h\right]s^{\mu}dx^{\sigma}/\left(i\hbar\right) + C\left[z_{\nu},h\right]s^{\mu}s^{\nu}\left(s_{\sigma}dx^{\sigma}\right)/\left(i\hbar\right) \\ &\quad + h\left[z_{\sigma},s^{\mu}\right]dx^{\sigma}/\left(i\hbar\right) + Ch\left[z_{\nu},s^{\mu}\right]s^{\nu}\left(s_{\sigma}dx^{\sigma}\right)/\left(i\hbar\right) \\ &= -\left(\hat{\partial}_{\nu}f\right)x^{\mu}dx^{\nu} - C\left(\hat{\partial}_{\kappa}f\right)s^{\kappa}x^{\mu}\left(s_{\nu}dx^{\nu}\right) - \left(\hat{\partial}_{\nu}h\right)s^{\mu}dx^{\nu} \\ &\quad - C\left(\hat{\partial}_{\kappa}h\right)s^{\mu}s^{\kappa}\left(s_{\nu}dx^{\nu}\right) - h\left(\hat{\partial}_{\nu}s^{\mu}\right)dx^{\nu} - Chs^{\mu}\left(s_{\nu}dx^{\nu}\right) \end{aligned}$$

Therefore:

$$\left(\partial_{\nu}f\right)x^{\mu}dx^{\nu} + \left(\partial_{\nu}h\right)s^{\mu}dx^{\nu}$$

$$
\begin{aligned}
& D\hat{x}^{\mu}-[\hat{Q},\hat{x}^{\mu}]/\left(i\hbar\right) \\
= \; & -\Gamma^{\rho}{}_{\sigma\nu}s^{\sigma}\left(\hat{\partial}_{\rho}f\right)x^{\mu}dx^{\nu}+fdx^{\mu}-\Gamma^{\rho}{}_{\sigma\nu}s^{\sigma}\left(\hat{\partial}_{\rho}h\right)s^{\mu}dx^{\nu}-hdx^{\nu}\Gamma^{\mu}{}_{\sigma\nu}s^{\sigma} \\
& -C\left(\hat{\partial}_{\kappa}f\right)s^{\kappa}x^{\mu}\left(s_{\nu}dx^{\nu}\right)-C\left(\hat{\partial}_{\kappa}h\right)s^{\mu}s^{\kappa}\left(s_{\nu}dx^{\nu}\right)-hdx^{\mu}-Chs^{\mu}\left(s_{\nu}dx^{\nu}\right) \\
& +\left(\partial_{\nu}f\right)x^{\mu}dx^{\nu}+\left(\partial_{\nu}h\right)s^{\mu}dx^{\nu}-\left(\hat{\partial}_{\nu}f\right)x^{\mu}dx^{\nu}-\left(\hat{\partial}_{\nu}h\right)s^{\mu}dx^{\nu} \\
= \; & \left(\left(-C\left(x^{\nu}+s^{\nu}\right)\hat{\partial}_{\nu}f-Ch\right)\left(s_{\nu}dx^{\nu}\right)+\left(\partial_{\nu}f-\hat{\partial}_{\nu}f\right)dx^{\nu}\right)x^{\mu} \\
& +\left(\left(-C\left(x^{\nu}+s^{\nu}\right)\hat{\partial}_{\nu}h-Ch\right)\left(s_{\nu}dx^{\nu}\right)+\left(\partial_{\nu}h-\hat{\partial}_{\nu}h\right)dx^{\nu}\right)s^{\mu} \\
& +\left(f-h\right)dx^{\mu}
\end{aligned}
$$

To simplify further we let f only be a matrix-valued function of $u=s_{\mu}s^{\mu}$:

$$
\begin{aligned}
D\hat{x}^{\mu}-[\hat{Q},\hat{x}^{\mu}]/\left(i\hbar\right) \; = \; & \left(\left(-2Cuf'-Ch\right)-2f'\right)\left(s_{\nu}dx^{\nu}\right)x^{\mu} \\
& +\left(\left(-2Cuh'-Ch\right)-2h'\right)\left(s_{\nu}dx^{\nu}\right)s^{\mu}+\left(f-h\right)dx^{\mu}
\end{aligned}
$$

where $\hat{\partial}_{\nu}f=2s_{\nu}f'$ and $f':=\partial f/\partial u$. It is then easily observed that a solution yield the two conditions:

$$f=h$$

$$-2\left(Cu+1\right)f'-Cf=0$$

which has the solution:

$$f=\frac{B}{\sqrt{\left(Cu+1\right)}}$$

where B is an arbitrary constant. By putting this into (C.7) we get:

$$\hat{x}^{\mu}=\left(x^{\mu}+s^{\mu}\right)\frac{B}{\sqrt{\left(Cu+1\right)}}$$

However, we still need to impose the condition $\sigma\left(\hat{x}^{\mu}\right)=x^{\mu}$ in the original condition which determines B:

$$x^{\mu}=\sigma\left(\hat{x}^{\mu}\right)=Bx^{\mu}\implies B=1$$

Therefore:

$$\hat{x}^{\mu}=\left(x^{\mu}+s^{\mu}\right)\frac{1}{\sqrt{\left(Cu+1\right)}}$$

which is the solution in (6.19) (and (C.5)).

QED.

It is easily observed that:

$$\hat{x}_{\mu}\hat{x}^{\mu}=1/C$$

C.2.2 Proof of (6.20)

We start with the ansatz:

$$\hat{p}_\mu = z_\nu s^\nu x_\mu f(u) + z_\mu h(u) + i\hbar B\hat{x}_\mu$$

where B is an arbitrary constant and require that the condition (C.4) is satisfied:

$$\left(D - \hat{D}\right)\hat{p}_\mu = D\hat{p}_\mu - [\hat{Q}, \hat{p}_\mu]/(i\hbar) = 0 \quad , \quad \sigma(\hat{p}_\mu) = p_\mu \tag{C.8}$$

where $\hat{Q}$ is:

$$\hat{Q} = (s^\mu \alpha_\mu - z_\mu dx^\mu) - C(z_\nu s^\nu)(s_\mu dx^\mu) + \frac{C}{3}\left((p_\nu dx^\nu)u - (p_\nu s^\nu)(s_\mu dx^\mu)\right)$$

where $u = s_\mu s^\mu$ and $z_\mu = p_\mu + k_\mu$ (see (6.16)).

Proof of (6.20):

We know that $\left(D - \hat{D}\right)\hat{x}_\mu = 0$ where $\hat{x}_\mu$ is given by the formula in equation (6.19). Therefore $\left(D - \hat{D}\right)\hat{p}_\mu = 0$ iff $\left(D - \hat{D}\right)\hat{v}_\mu = 0$ where $\hat{v}_\mu := (\hat{p}_\mu - i\hbar B\hat{x}_\mu)$. So we compute

$$D\hat{v}_\mu = (Dz_\nu)s^\nu x_\mu f + z_\nu(Ds^\nu)x_\mu f + z_\nu s^\nu(Dx_\mu)f + z_\nu s^\nu x_\mu Df + (Dz_\mu)h + z_\mu Dh \tag{C.9}$$

calculating some useful quantities:

$$Ds^\nu = -\Gamma^\nu{}_{\rho\sigma}dx^\sigma s^\rho$$

$$\begin{aligned} Dz_\mu &= dp_\mu - \frac{4}{3}R^\psi{}_{(\mu\sigma)\beta}dx^\beta s^\sigma p_\psi + \Gamma^\nu{}_{\mu\sigma}dx^\sigma k_\nu \\ &= \alpha_\mu - \frac{4}{3}p_\psi R^\psi{}_{(\mu\sigma)\rho}dx^\rho s^\sigma + z_\rho\Gamma^\rho{}_{\mu\sigma}dx^\sigma \end{aligned}$$

$$D\otimes\alpha_\mu = \Theta^B\otimes D_B\alpha_\mu := -\frac{4}{3}R^\psi{}_{(\mu\sigma)\beta}p_\psi dx^\beta\otimes dx^\sigma + \Gamma^\nu{}_{\mu\sigma}dx^\sigma\otimes\alpha_\nu$$

Putting this back into (C.9):

$$\begin{aligned} D\hat{v}_\mu \;=\; & \left(\alpha_\nu - \frac{4}{3}p_\psi R^\psi{}_{(\nu\sigma)\rho}dx^\rho s^\sigma + z_\rho\Gamma^\rho{}_{\nu\sigma}dx^\sigma\right)s^\nu x_\mu f \\ & - z_\nu\Gamma^\nu{}_{\rho\sigma}dx^\sigma s^\rho x_\mu f + z_\nu s^\nu(dx_\mu)f + z_\nu s^\nu x_\mu dx^\sigma\left(\partial_\sigma f - \Gamma^\rho{}_{\psi\sigma}s^\psi\hat{\partial}_\rho f\right) \\ & + \left(\alpha_\mu - \frac{4}{3}p_\psi R^\psi{}_{(\mu\sigma)\rho}dx^\rho s^\sigma + z_\rho\Gamma^\rho{}_{\mu\sigma}dx^\sigma\right)h + z_\mu dx^\sigma\left(\partial_\sigma h - \Gamma^\rho{}_{\psi\sigma}s^\psi\hat{\partial}_\rho h\right) \end{aligned}$$

$$\Gamma^{\rho}{}_{\mu\sigma} = Cx^{\rho} g_{\mu\sigma} - 2Cx_{(\mu}\left(\delta^{\rho}_{\sigma)} - Cx_{\sigma)}x^{\rho}\right)$$

$$\begin{aligned} 4R^{\psi}{}_{(\mu\sigma)\rho} &= -2C\left(\delta^{\psi}_{[\sigma} - Cx_{[\sigma}x^{\psi}\right) g_{\rho]\mu} - 2C\left(\delta^{\psi}_{[\mu} - Cx_{[\mu}x^{\psi}\right) g_{\rho]\sigma} \\ &= -C\left(\left(\delta^{\psi}_{\sigma} - Cx_{\sigma}x^{\psi}\right) g_{\rho\mu} - 2\left(\delta^{\psi}_{\rho} - Cx_{\rho}x^{\psi}\right) g_{\sigma\mu} + \left(\delta^{\psi}_{\mu} - Cx_{\mu}x^{\psi}\right) g_{\rho\sigma}\right) \end{aligned}$$

$$\begin{aligned} D\hat{v}_{\mu} &= \alpha_{\nu} s^{\nu} x_{\mu} f + z_{\rho}\left(Cx^{\rho} g_{\nu\sigma} - 2Cx_{(\nu}\left(\delta^{\rho}_{\sigma)} - Cx_{\sigma)}x^{\rho}\right)\right) dx^{\sigma} s^{\nu} x_{\mu} f \\ &\quad + \frac{C}{3} p_{\psi} \begin{pmatrix} \left(\delta^{\psi}_{\sigma} - Cx_{\sigma}x^{\psi}\right) g_{\rho\nu} - 2\left(\delta^{\psi}_{\rho} - Cx_{\rho}x^{\psi}\right) g_{\sigma\nu} \\ + \left(\delta^{\psi}_{\nu} - Cx_{\nu}x^{\psi}\right) g_{\rho\sigma} \end{pmatrix} dx^{\rho} s^{\sigma} s^{\nu} x_{\mu} f \\ &\quad - z_{\nu}\left(Cx^{\nu} g_{\rho\sigma} - 2Cx_{(\rho}\left(\delta^{\nu}_{\sigma)} - Cx_{\sigma)}x^{\nu}\right)\right) dx^{\sigma} s^{\rho} x_{\mu} f + z_{\nu} s^{\nu}\left(dx_{\mu}\right) f \\ &\quad + z_{\nu} s^{\nu} x_{\mu} dx^{\sigma}\left(\partial_{\sigma} f - Cx^{\rho} g_{\psi\sigma} s^{\psi} \hat{\partial}_{\rho} f\right) \\ &\quad + \alpha_{\mu} h + \frac{C}{3} p_{\psi} \begin{pmatrix} \left(\delta^{\psi}_{\sigma} - Cx_{\sigma}x^{\psi}\right) g_{\rho\mu} - 2\left(\delta^{\psi}_{\rho} - Cx_{\rho}x^{\psi}\right) g_{\sigma\mu} \\ + \left(\delta^{\psi}_{\mu} - Cx_{\mu}x^{\psi}\right) g_{\rho\sigma} \end{pmatrix} dx^{\rho} s^{\sigma} h \\ &\quad + z_{\rho}\left(Cx^{\rho} g_{\mu\sigma} - 2Cx_{(\mu}\left(\delta^{\rho}_{\sigma)} - Cx_{\sigma)}x^{\rho}\right)\right) dx^{\sigma} h \\ &\quad + z_{\mu} dx^{\sigma}\left(\partial_{\sigma} h - Cx^{\rho} g_{\psi\sigma} s^{\psi} \hat{\partial}_{\rho} h\right) \end{aligned}$$

Computing and grouping terms appropriately:

$$\begin{aligned} D\hat{v}_{\mu} &= \left((s\cdot\alpha) f + \frac{2C}{3}(p\cdot s)(s\cdot dx) f - \frac{2C}{3}(p\cdot dx) u f - C(z\cdot dx) h\right) x_{\mu} \\ &\quad + \left((z\cdot s) f + \frac{C}{3}(p\cdot s) h\right) dx_{\mu} + \alpha_{\mu} h - \frac{2C}{3}(p\cdot dx) h s_{\mu} + \frac{C}{3} p_{\mu} h (s\cdot dx) \end{aligned}$$

where we remember that $x^{\mu} p_{\mu} = 0$ and $\hat{\partial}_{\rho} f = 2 s_{\rho} f'$ where $f' := \partial f / \partial u$ and $u = s^{\mu} s_{\mu}$.

Computing $-[\hat{Q}, \hat{v}_{\mu}]/(i\hbar)$:

$$\hat{Q} = \left(s^{\mu}\alpha_{\mu} - z_{\mu} dx^{\mu}\right) - C\left(z_{\nu} s^{\nu}\right)\left(s_{\mu} dx^{\mu}\right) + \frac{C}{3}\left(\left(p_{\nu} s^{\nu}\right)\left(s_{\mu} dx^{\mu}\right) - \left(p_{\nu} dx^{\nu}\right) u\right)$$

$$\hat{p}_{\mu} = z_{\nu} s^{\nu} x_{\mu} f(u) + z_{\mu} h(u) + B\hat{x}_{\mu}$$

$$
\begin{aligned}
-[\hat{Q},\hat{v}_\mu]/\left(i\hbar\right) \;=\;& \left[-s^\rho\alpha_\rho, z_\mu h\right]/\left(i\hbar\right)+\left[-s^\rho\alpha_\rho, z_\nu s^\nu x_\mu f\right]/\left(i\hbar\right)\\
&+\left[z_\rho dx^\rho, z_\nu s^\nu x_\mu f\right]/\left(i\hbar\right)+\left[z_\rho dx^\rho, z_\mu h\right]/\left(i\hbar\right)\\
&+C\left[\left(z_\sigma s^\sigma\right)\left(s_\rho dx^\rho\right), z_\nu s^\nu x_\mu f\right]/\left(i\hbar\right)+C\left[\left(z_\sigma s^\sigma\right)\left(s_\rho dx^\rho\right), z_\mu h\right]/\left(i\hbar\right)\\
&-\frac{C}{3}\left[\left(p_\sigma s^\sigma\right)\left(s_\rho dx^\rho\right), z_\nu s^\nu x_\mu f\right]/\left(i\hbar\right)-\frac{C}{3}\left[\left(p_\sigma s^\sigma\right)\left(s_\rho dx^\rho\right), z_\mu h\right]/\left(i\hbar\right)\\
&+\frac{C}{3}\left[\left(p_\sigma dx^\sigma\right)u, z_\nu s^\nu x_\mu f\right]/\left(i\hbar\right)+\frac{C}{3}\left[\left(p_\sigma dx^\sigma\right)u, z_\mu h\right]/\left(i\hbar\right)
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\alpha_\rho\left[s^\rho, z_\mu\right]h/\left(i\hbar\right)-\alpha_\rho\left[s^\rho, z_\nu\right]s^\nu x_\mu f/\left(i\hbar\right)-z_\nu\left[s^\nu, z_\rho\right]f x_\mu dx^\rho/\left(i\hbar\right)\\
&-z_\nu s^\nu\left[f, z_\rho\right]x_\mu dx^\rho/\left(i\hbar\right)-z_\mu\left[h, z_\rho\right]dx^\rho/\left(i\hbar\right)\\
&+Cz_\sigma\left[s^\sigma, z_\nu\right]s_\rho s^\nu dx^\rho x_\mu f/\left(i\hbar\right)+Cz_\sigma s^\sigma\left[s_\rho, z_\nu\right]s^\nu dx^\rho x_\mu f/\left(i\hbar\right)\\
&-Cz_\nu\left[s^\nu, z_\sigma\right]s^\sigma\left(s_\rho dx^\rho\right)x_\mu f/\left(i\hbar\right)\\
&-Cz_\nu s^\nu x_\mu\left[f, z_\sigma\right]s^\sigma s_\rho dx^\rho/\left(i\hbar\right)+Cz_\sigma\left[s^\sigma, z_\mu\right]s_\rho dx^\rho h/\left(i\hbar\right)+Cz_\sigma s^\sigma\left[s_\rho, z_\mu\right]dx^\rho h/\left(i\hbar\right)\\
&-Cz_\mu\left[h, z_\sigma\right]s^\sigma s_\rho dx^\rho/\left(i\hbar\right)-\frac{C}{3}p_\sigma\left[s^\sigma, z_\nu\right]s_\rho s^\nu dx^\rho x_\mu f/\left(i\hbar\right)\\
&-\frac{C}{3}p_\sigma s^\sigma\left[s_\rho, z_\nu\right]s^\nu dx^\rho x_\mu f/\left(i\hbar\right)\\
&-\frac{C}{3}p_\sigma\left[s^\sigma, z_\mu\right]s_\rho h dx^\rho/\left(i\hbar\right)-\frac{C}{3}p_\sigma s^\sigma\left[s_\rho, z_\mu\right]h dx^\rho/\left(i\hbar\right)\\
&+\frac{C}{3}p_\sigma dx^\sigma\left[u, z_\nu\right]s^\nu x_\mu f/\left(i\hbar\right)+\frac{C}{3}p_\sigma dx^\sigma\left[u, z_\mu\right]h/\left(i\hbar\right)
\end{aligned}
$$

$$
\begin{aligned}
=\;& -\alpha_\mu h-\left(s\cdot\alpha\right)x_\mu f-\left(z\cdot dx\right)f x_\mu-\left(z\cdot s\right)\left(\hat{\partial}_\rho f\right)x_\mu dx^\rho-z_\mu\left(\hat{\partial}_\rho h\right)dx^\rho\\
&+C\left(z\cdot s\right)\left(s\cdot dx\right)f x_\mu+C\left(z\cdot s\right)\left(s\cdot dx\right)x_\mu f-C\left(z\cdot s\right)\left(s\cdot dx\right)x_\mu f\\
&-C\left(z\cdot s\right)\left(s\cdot dx\right)s^\sigma\left(\hat{\partial}_\sigma f\right)x_\mu+C\left(z_\mu-CAx_\mu\right)\left(s\cdot dx\right)h+C\left(z\cdot s\right)h dx_\mu\\
&-Cz_\mu s^\sigma\left(\hat{\partial}_\sigma h\right)\left(s\cdot dx\right)-\frac{C}{3}\left(p\cdot s\right)\left(s\cdot dx\right)f x_\mu-\frac{C}{3}\left(p\cdot s\right)\left(s\cdot dx\right)x_\mu f\\
&-\frac{C}{3}\left(p_\mu-CAx_\mu\right)\left(s\cdot dx\right)h-\frac{C}{3}\left(p\cdot s\right)h dx_\mu\\
&+\frac{2C}{3}\left(p\cdot dx\right)u f x_\mu+\frac{2C}{3}\left(p\cdot dx\right)h s_\mu
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\alpha_\mu h - (s\cdot\alpha)\, x_\mu f - (z\cdot dx)\, f x_\mu - 2\,(z\cdot s)\, f'\,(s\cdot dx)\, x_\mu - 2 z_\mu h'\,(s\cdot dx) \\
& + C\,(z\cdot s)\,(s\cdot dx)\, f x_\mu + C\,(z\cdot s)\,(s\cdot dx)\, x_\mu f - C\,(z\cdot s)\,(s\cdot dx)\, x_\mu f \\
& - 2C\,(z\cdot s)\,(s\cdot dx)\, u f' x_\mu + C\,(z_\mu - C A x_\mu)\,(s\cdot dx)\, h + C\,(z\cdot s)\, h\, dx_\mu \\
& - 2C z_\mu u h'\,(s\cdot dx) - \frac{C}{3}\,(p\cdot s)\,(s\cdot dx)\, f x_\mu - \frac{C}{3}\,(p\cdot s)\,(s\cdot dx)\, x_\mu f \\
& - \frac{C}{3} p_\mu\,(s\cdot dx)\, h - \frac{C}{3}\,(p\cdot s)\, h\, dx_\mu + \frac{2C}{3}\,(p\cdot dx)\, u f x_\mu + \frac{2C}{3}\,(p\cdot dx)\, h s_\mu
\end{aligned}
$$

where we note that $z_\mu x^\mu = p_\mu x^\mu = A$ and knowing that $\hat{\partial}_\rho f = 2 s_\rho f'$ ($f' := \partial f/\partial u$). We now group terms appropriately:

$$
\begin{aligned}
-[\hat{Q}, \hat{v}_\mu]/\,(i\hbar) \quad = \quad & \begin{pmatrix} -(s\cdot\alpha)\, f - (z\cdot dx)\, f + \frac{2C}{3}\,(p\cdot dx)\, u f \\ +(z\cdot s)\,(Cf - 2\,(Cu+1)\, f')\,(s\cdot dx) - \frac{2C}{3} f\,(p\cdot s)\,(s\cdot dx) \end{pmatrix} x_\mu \\
& + \left(C\,(z\cdot s)\, h - \frac{C}{3}\,(p\cdot s)\, h \right) dx_\mu + z_\mu\,(-2\,(Cu+1)\, h' + Ch)\,(s\cdot dx) \\
& - \frac{C}{3} p_\mu\,(s\cdot dx)\, h + \frac{2C}{3}\,(p\cdot dx)\, h s_\mu - \alpha_\mu h
\end{aligned}
$$

Putting it all together to get $D\hat{v}_\mu - [\hat{Q}, \hat{v}_\mu]/\,(i\hbar)$:

$$
\begin{aligned}
D\hat{v}_\mu - [\hat{Q}, \hat{v}_\mu]/\,(i\hbar) \quad = \quad & \left(-(z\cdot dx)\,(Ch + f) + (z\cdot s)\,(Cf - 2\,(Cu+1)\, f')\,(s\cdot dx) \right) x_\mu \\
& + \left((z\cdot s)\,(f + Ch) \right) dx_\mu + z_\mu\,(-2\,(Cu+1)\, h' + Ch)\,(s\cdot dx)
\end{aligned}
$$

Therefore the condition $D\hat{v}_\mu - [\hat{Q}, \hat{v}_\mu]/\,(i\hbar) = 0$ yields the two conditions:

$$
f + Ch = 0
$$

$$
Cf - 2\,(Cu+1)\, f' = 0
$$

which has the solution:

$$
\implies f' = \frac{fC}{2\,(Cu+1)} \implies f = CA\sqrt{Cu+1} \implies h = A\sqrt{Cu+1}
$$

where A is a constant determined by the condition in (C.8):

$$
p_\mu = \sigma\,(\hat{p}_\mu) = \sigma\,(-z_\mu A) = -p_\mu A \implies A = -1
$$

so:

$$\hat{p}_\mu = (z_\mu - z_\nu s^\nu x_\mu C)\sqrt{Cu+1} + i\hbar B\hat{x}_\mu$$

To fix B we want to require that $\hat{x}^\mu \hat{p}_\mu = 0$. We use the fact that $\hat{x}_\mu \hat{x}^\mu = 1/C$ and that $[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\,(\delta^\mu_\nu - C\hat{x}^\mu x_\nu)$ (we will prove this later):

$$0 = \hat{x}^\mu \hat{p}_\mu = \hat{p}_\mu \hat{x}^\mu + i\hbar n = i\hbar B/C + i\hbar n \implies B = Cn$$

Thus:

$$\hat{p}_\mu = (-Cz_\nu s^\nu x_\mu + z_\mu)\sqrt{Cu+1} - iC\hbar n\hat{x}_\mu$$

which is the formula in (6.20).

QED.

C.3 PROOF OF THE COMMUTATORS IN (6.25)

In this appendix section we verify the commutators in:

Obviously:

$$[\hat{x}^\mu, \hat{x}^\nu] = 0 \tag{C.10}$$

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\,(\delta^\mu_\nu - C\hat{x}_\nu \hat{x}^\mu) \tag{C.11}$$

$$[\hat{p}_\mu, \hat{p}_\nu] = -2i\hbar\hat{x}_{[\mu}\hat{p}_{\nu]} \tag{C.12}$$

and (6.25) starting with the formulas in (6.22) and (6.23).

Proof of (C.11)**:**

$$[\hat{x}^\mu, \hat{p}_\nu]\,/\,(i\hbar) = ([\hat{x}^\mu, z_\nu] - C\,[\hat{x}^\mu, z_\rho]\,s^\rho x_\nu)\sqrt{Cu+1}/\,(i\hbar)$$

First we compute $[\hat{x}^\mu, z_\nu]\,/\,(i\hbar)$:

$$
\begin{aligned}
[\hat{x}^\mu, z_\nu]\,/\,(i\hbar) &= [s^\mu, z_\nu]\,\frac{1}{i\hbar\sqrt{Cu+1}} + (x^\mu + s^\mu)\left[\frac{1}{i\hbar\sqrt{Cu+1}}, z_\nu\right] \\
&= (\delta^\mu_\nu - Cx^\mu x_\nu)\,\frac{1}{\sqrt{Cu+1}} + (x^\mu + s^\mu)\,\hat{\partial}_\nu\left(\frac{1}{\sqrt{Cu+1}}\right) \\
&= (\delta^\mu_\nu - Cx^\mu x_\nu)\,\frac{1}{\sqrt{Cu+1}} + (x^\mu + s^\mu)\left(\frac{-Cs_\nu}{(Cu+1)^{3/2}}\right) \\
&= (\delta^\mu_\nu - Cx^\mu x_\nu)\,\frac{1}{\sqrt{Cu+1}} - \frac{Cs_\nu\hat{x}^\mu}{Cu+1}
\end{aligned}
$$

and putting this back into the commutator $[\hat{x}^\mu, \hat{p}_\nu]\,/\,(i\hbar)$:

$$
\begin{aligned}
[\hat{x}^\mu, \hat{p}_\nu]\,/\,(i\hbar) &= \begin{pmatrix} (\delta^\mu_\nu - Cx^\mu x_\nu)\,\frac{1}{\sqrt{Cu+1}} - \frac{Cs_\nu\hat{x}^\mu}{Cu+1} \\ -C\left(\left(\delta^\mu_\rho - Cx^\mu x_\rho\right)\frac{1}{\sqrt{Cu+1}} - \frac{Cs_\rho\hat{x}^\mu}{Cu+1}\right)s^\rho x_\nu \end{pmatrix}\sqrt{Cu+1} \\
&= \delta^\mu_\nu - Cx^\mu x_\nu - \frac{Cs_\nu\hat{x}^\mu}{\sqrt{Cu+1}} - C\left(s^\mu - \frac{Cu\hat{x}^\mu}{\sqrt{Cu+1}}\right)x_\nu \\
&= \delta^\mu_\nu - Cx^\mu x_\nu - \frac{Cs_\nu\hat{x}^\mu}{\sqrt{Cu+1}} - C\left(\frac{1}{\sqrt{Cu+1}}x_\nu\right)\hat{x}^\mu + Cx^\mu x_\nu \\
&= \delta^\mu_\nu - Cx^\mu x_\nu - C\hat{x}_\nu\hat{x}^\mu + Cx^\mu x_\nu = \delta^\mu_\nu - C\hat{x}_\nu\hat{x}^\mu
\end{aligned}
$$

$$
[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\,(\delta^\mu_\nu - C\hat{x}_\nu\hat{x}^\mu)
$$

which is the correct one in (C.11).

QED.

Proof of (C.12)**:**

$$\begin{aligned}
& [\hat{p}_\mu, \hat{p}_\nu] / (i\hbar) \\
= \; & \left[\left((z_\mu - C z_\rho s^\rho x_\mu)\sqrt{Cu+1} - C(i\hbar n + A)\hat{x}_\mu\right), (z_\nu - C z_\sigma s^\sigma x_\nu)\sqrt{Cu+1}\right] / (i\hbar) \\
& -C(i\hbar n + A)\left[\left((z_\mu - C z_\rho s^\rho x_\mu)\sqrt{Cu+1} - C(i\hbar n + A)\hat{x}_\mu\right), \hat{x}_\nu\right] / (i\hbar) \\
= \; & \left[(z_\mu - C z_\rho s^\rho x_\mu)\sqrt{Cu+1}, (z_\nu - C z_\sigma s^\sigma x_\nu)\sqrt{Cu+1}\right] / (i\hbar) \\
& -C(i\hbar n + A)\left[\hat{x}_\mu, (z_\nu - C z_\sigma s^\sigma x_\nu)\sqrt{Cu+1}\right] / (i\hbar) \\
& -C(i\hbar n + A)\left[(z_\mu - C z_\rho s^\rho x_\mu)\sqrt{Cu+1}, \hat{x}_\nu\right] / (i\hbar) \\
= \; & \left[(z_\mu - C z_\rho s^\rho x_\mu)\sqrt{Cu+1}, (z_\nu - C z_\sigma s^\sigma x_\nu)\sqrt{Cu+1}\right] / (i\hbar) \\
& -C(i\hbar n + A)\left([\hat{x}_\mu, \hat{p}_\nu] - [\hat{p}_\mu, \hat{x}_\nu]\right) / (i\hbar) \\
= \; & \left[(z_\mu - C z_\rho s^\rho x_\mu)\sqrt{Cu+1}, (z_\nu - C z_\sigma s^\sigma x_\nu)\sqrt{Cu+1}\right] / (i\hbar) \\
= \; & (z_\mu - C z_\rho s^\rho x_\mu)\left[\sqrt{Cu+1}, (z_\nu - C z_\sigma s^\sigma x_\nu)\right]\sqrt{Cu+1} / (i\hbar) \\
& + \left[(z_\mu - C z_\rho s^\rho x_\mu), (z_\nu - C z_\sigma s^\sigma x_\nu)\right](Cu+1) / (i\hbar) \\
& + (z_\nu - C z_\sigma s^\sigma x_\nu)\left[(z_\mu - C z_\rho s^\rho x_\mu), \sqrt{Cu+1}\right]\sqrt{Cu+1} / (i\hbar) \\
= \; & \underbrace{(z_\mu - C z_\rho s^\rho x_\mu)\left[\sqrt{Cu+1}, (z_\nu - C z_\sigma s^\sigma x_\nu)\right]\sqrt{Cu+1} / (i\hbar)}_{(A)} \\
& + \underbrace{\left[(z_\mu - C z_\rho s^\rho x_\mu), (z_\nu - C z_\sigma s^\sigma x_\nu)\right](Cu+1) / (i\hbar)}_{(B)} \\
& + (z_\nu - C z_\sigma s^\sigma x_\nu)\left[(z_\mu - C z_\rho s^\rho x_\mu), \sqrt{Cu+1}\right]\sqrt{Cu+1} / (i\hbar)
\end{aligned}$$

We compute:

$$\begin{aligned}
(A) \; = \; & (z_\mu - C z_\rho s^\rho x_\mu)\left(\left[\sqrt{Cu+1}, z_\nu\right] - C\left[\sqrt{Cu+1}, z_\sigma\right] s^\sigma x_\nu\right)\sqrt{Cu+1} / (i\hbar) \\
= \; & (z_\mu - C z_\rho s^\rho x_\mu)\left(\frac{C s_\nu}{\sqrt{Cu+1}} - \frac{C^2 u}{\sqrt{Cu+1}} x_\nu\right)\sqrt{Cu+1} \\
= \; & (\hat{p}_\mu + C(i\hbar n + A)\hat{x}_\mu)\left(\frac{C s_\nu}{\sqrt{Cu+1}} - \frac{C^2 u}{\sqrt{Cu+1}} x_\nu\right)
\end{aligned}$$

and:

$$\begin{aligned}
(B) &= -Cz_\rho\left[s^\rho, z_\nu\right]x_\mu\left(Cu+1\right)/\left(i\hbar\right) + Cz_\sigma\left[s^\sigma, z_\mu\right]x_\nu\left(Cu+1\right)/\left(i\hbar\right) \\
&= 2Cz_{[\mu}x_{\nu]}\left(Cu+1\right) = Cz_\mu x_\nu\left(Cu+1\right) - Cz_\nu x_\mu\left(Cu+1\right) \\
&= C\left(z_\mu - z_\rho s^\rho x_\mu\right)x_\nu\left(Cu+1\right) - C\left(z_\nu - z_\rho s^\rho x_\nu\right)x_\mu\left(Cu+1\right) \\
&= C\left(\hat{p}_\mu + C\left(i\hbar n + A\right)\hat{x}_\mu\right)x_\nu\sqrt{Cu+1} - C\left(\hat{p}_\nu + C\left(i\hbar n + A\right)\hat{x}_\nu\right)x_\mu\sqrt{Cu+1}
\end{aligned}$$

Plugging them back in:

$$\begin{aligned}
\left[\hat{p}_\mu, \hat{p}_\nu\right]/\left(i\hbar\right) &= \left(\hat{p}_\mu + C\left(i\hbar n + A\right)\hat{x}_\mu\right)\left(\frac{Cs_\nu}{\sqrt{Cu+1}} - \frac{C^2u}{\sqrt{Cu+1}}x_\nu + Cx_\nu\sqrt{Cu+1}\right) \\
&\quad - \left(\hat{p}_\nu + C\left(i\hbar n + A\right)\hat{x}_\nu\right)\left(\frac{Cs_\mu}{\sqrt{Cu+1}} - \frac{C^2u}{\sqrt{Cu+1}}x_\mu - Cx_\mu\sqrt{Cu+1}\right)
\end{aligned}$$

$$\left[\hat{p}_\mu, \hat{p}_\nu\right]/\left(i\hbar\right) = \left(\hat{p}_\mu + C\left(i\hbar n + A\right)\hat{x}_\mu\right)\hat{x}_\nu - \left(\hat{p}_\nu + C\left(i\hbar n + A\right)\hat{x}_\nu\right)\hat{x}_\mu = 2\hat{p}_{[\mu}\hat{x}_{\nu]} = -2\hat{x}_{[\mu}\hat{p}_{\nu]}$$

so:

$$\left[\hat{p}_\mu, \hat{p}_\nu\right] = -2i\hbar\hat{x}_{[\mu}\hat{p}_{\nu]}$$

which is the one in (C.12).

QED.

C.4 PROOF OF THE COMMUTATORS IN (6.27)

Knowing the formulas in (6.26) and (6.25) we prove that the commutators are (6.27):

$$\left[\hat{x}_\mu, \hat{M}_{\nu\rho}\right] = i\hbar\hat{x}_{[\nu}\eta_{\rho]\mu} \tag{C.13}$$

$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = i\hbar\left(\hat{M}_{\sigma[\mu}\eta_{\nu]\rho} - \hat{M}_{\rho[\mu}\eta_{\nu]\sigma}\right) \tag{C.14}$$

Proof of (C.13):

$$\left[\hat{x}_\mu, \hat{M}_{\nu\rho}\right]/\left(i\hbar\right) = -\hat{x}_{[\nu}\left[\hat{p}_{\rho]}, \hat{x}_\mu\right] = \hat{x}_{[\nu}\left(\eta_{\rho]\mu} - C\hat{x}_{\rho]}\hat{x}_\mu\right) = \hat{x}_{[\nu}\eta_{\rho]\mu}$$

QED.

Proof of (C.14)**:**

$$\begin{aligned}
\left[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}\right]/(i\hbar) &= \left[\hat{x}_{[\mu}\hat{p}_{\nu]},\hat{x}_{[\rho}\hat{p}_{\sigma]}\right]/(i\hbar)\\
&= \left[\hat{x}_{[\mu}\hat{p}_{\nu]},\hat{x}_{[\rho}\right]\hat{p}_{\sigma]}/(i\hbar)-\hat{x}_{[\rho}\left[\hat{p}_{\sigma]},\hat{x}_{[\mu}\hat{p}_{\nu]}\right]/(i\hbar)\\
&= \hat{x}_{[\mu}\left[\hat{p}_{\nu]},\hat{x}_{[\rho}\right]\hat{p}_{\sigma]}/(i\hbar)\\
&\quad+\begin{pmatrix} -\hat{x}_{[\rho}\left[\hat{p}_{\sigma]},\hat{x}_{[\mu}\right]\hat{p}_{\nu]}-\hat{x}_{\mu}\hat{x}_{\rho}\left[\hat{p}_{\sigma},\hat{p}_{\nu}\right]/4+\hat{x}_{\nu}\hat{x}_{\rho}\left[\hat{p}_{\sigma},\hat{p}_{\mu}\right]/4\\ +\hat{x}_{\mu}\hat{x}_{\sigma}\left[\hat{p}_{\rho},\hat{p}_{\nu}\right]/4-\hat{x}_{\nu}\hat{x}_{\sigma}\left[\hat{p}_{\rho},\hat{p}_{\mu}\right]/4\end{pmatrix}/(i\hbar)\\
&= -\hat{x}_{[\mu}\left(\eta_{\nu][\rho}-C\hat{x}_{\nu]}\hat{x}_{[\rho}\right)\hat{p}_{\sigma]}\\
&\quad+\begin{pmatrix} \hat{x}_{[\rho}\left(\eta_{\sigma][\mu}-C\hat{x}_{\sigma]}\hat{x}_{[\mu}\right)\hat{p}_{\nu]}-\hat{x}_{\mu}\hat{x}_{\rho}\hat{x}_{[\sigma}\hat{p}_{\nu]}/4+\hat{x}_{\nu}\hat{x}_{\rho}\hat{x}_{[\sigma}\hat{p}_{\mu]}/4\\ +\hat{x}_{\mu}\hat{x}_{\sigma}\hat{x}_{[\rho}\hat{p}_{\nu]}/4-\hat{x}_{\nu}\hat{x}_{\sigma}\hat{x}_{[\rho}\hat{p}_{\mu]}/4\end{pmatrix}\\
&= -\hat{x}_{[\mu}\eta_{\nu][\rho}\hat{p}_{\sigma]}+\hat{x}_{[\rho}\eta_{\sigma][\mu}\hat{p}_{\nu]}\\
&\quad-\hat{x}_{\mu}\hat{x}_{\rho}\hat{x}_{[\sigma}\hat{p}_{\nu]}/4+\hat{x}_{\nu}\hat{x}_{\rho}\hat{x}_{[\sigma}\hat{p}_{\mu]}/4+\hat{x}_{\mu}\hat{x}_{\sigma}\hat{x}_{[\rho}\hat{p}_{\nu]}/4-\hat{x}_{\nu}\hat{x}_{\sigma}\hat{x}_{[\rho}\hat{p}_{\mu]}/4\\
&= -\hat{x}_{[\mu}\eta_{\nu][\rho}\hat{p}_{\sigma]}+\hat{x}_{[\rho}\eta_{\sigma][\mu}\hat{p}_{\nu]}\\
&\quad-\hat{x}_{\mu}\hat{x}_{\rho}\hat{x}_{[\sigma}\hat{p}_{\nu]}/4+\hat{x}_{\nu}\hat{x}_{\rho}\hat{x}_{[\sigma}\hat{p}_{\mu]}/4+\hat{x}_{\mu}\hat{x}_{\sigma}\hat{x}_{[\rho}\hat{p}_{\nu]}/4-\hat{x}_{\nu}\hat{x}_{\sigma}\hat{x}_{[\rho}\hat{p}_{\mu]}/4
\end{aligned}$$

$$\begin{aligned}
4\left[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}\right]/(i\hbar) &= -\hat{x}_{\mu}\eta_{\nu\rho}\hat{p}_{\sigma}+\hat{x}_{\nu}\eta_{\mu\rho}\hat{p}_{\sigma}+\hat{x}_{\mu}\eta_{\nu\sigma}\hat{p}_{\rho}-\hat{x}_{\nu}\eta_{\mu\sigma}\hat{p}_{\rho}\\
&\quad+\hat{x}_{\rho}\eta_{\sigma\mu}\hat{p}_{\nu}-\hat{x}_{\sigma}\eta_{\rho\mu}\hat{p}_{\nu}-\hat{x}_{\rho}\eta_{\sigma\nu}\hat{p}_{\mu}+\hat{x}_{\sigma}\eta_{\rho\nu}\hat{p}_{\mu}\\
&\quad+\hat{x}_{\mu}\hat{x}_{\rho}\left(\hat{x}_{\nu}\hat{p}_{\sigma}-\hat{x}_{\sigma}\hat{p}_{\nu}\right)/2+\hat{x}_{\nu}\hat{x}_{\rho}\left(\hat{x}_{\sigma}\hat{p}_{\mu}-\hat{x}_{\mu}\hat{p}_{\sigma}\right)/2\\
&\quad+\hat{x}_{\mu}\hat{x}_{\sigma}\left(\hat{x}_{\rho}\hat{p}_{\nu}-\hat{x}_{\nu}\hat{p}_{\rho}\right)/2-\hat{x}_{\nu}\hat{x}_{\sigma}\left(\hat{x}_{\rho}\hat{p}_{\mu}-\hat{x}_{\mu}\hat{p}_{\rho}\right)/2
\end{aligned}$$

$$\begin{aligned}
&= -\hat{x}_{\mu}\eta_{\nu\rho}\hat{p}_{\sigma}+\hat{x}_{\nu}\eta_{\mu\rho}\hat{p}_{\sigma}+\hat{x}_{\mu}\eta_{\nu\sigma}\hat{p}_{\rho}-\hat{x}_{\nu}\eta_{\mu\sigma}\hat{p}_{\rho}\\
&\quad+\hat{x}_{\rho}\eta_{\sigma\mu}\hat{p}_{\nu}-\hat{x}_{\sigma}\eta_{\rho\mu}\hat{p}_{\nu}-\hat{x}_{\rho}\eta_{\sigma\nu}\hat{p}_{\mu}+\hat{x}_{\sigma}\eta_{\rho\nu}\hat{p}_{\mu}\\
&= \hat{x}_{\rho}\hat{p}_{\mu}\eta_{\nu\sigma}-\hat{x}_{\rho}\hat{p}_{\nu}\eta_{\mu\sigma}-\eta_{\sigma\nu}\hat{x}_{\mu}\hat{p}_{\rho}+\eta_{\sigma\mu}\hat{x}_{\nu}\hat{p}_{\rho}\\
&\quad-\hat{x}_{\sigma}\hat{p}_{\mu}\eta_{\nu\rho}+\hat{x}_{\sigma}\hat{p}_{\nu}\eta_{\mu\rho}+\eta_{\rho\nu}\hat{x}_{\mu}\hat{p}_{\sigma}-\eta_{\rho\mu}\hat{x}_{\nu}\hat{p}_{\sigma}\\
&= 4\left(\hat{M}_{\sigma[\mu}\eta_{\nu]\rho}-\hat{M}_{\rho[\mu}\eta_{\nu]\sigma}\right)
\end{aligned}$$

so that:

$$\left[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}\right]=i\hbar\left(\hat{M}_{\sigma[\mu}\eta_{\nu]\rho}-\hat{M}_{\rho[\mu}\eta_{\nu]\sigma}\right)$$

which is (C.14)

QED.

C.5 PROOF OF THE COMMUTATORS (6.30)

We want to show that the commutators in (6.27):

$$[\hat{x}^{\mu},\hat{x}^{\nu}]=0$$

$$\left[\hat{x}_{\mu},\hat{M}_{\nu\rho}\right]=i\hbar\hat{x}_{[\nu}\eta_{\rho]\mu}$$

$$\left[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}\right]=i\hbar\left(\hat{M}_{\sigma[\mu}\eta_{\nu]\rho}-\hat{M}_{\rho[\mu}\eta_{\nu]\sigma}\right)$$

are equivalent to(6.30):

$$\left[\hat{M}_{\mu'\nu'},\hat{M}_{\rho'\sigma'}\right]=i\hbar\left(\hat{M}_{\rho'[\mu'}\eta_{\nu']\sigma'}-\hat{M}_{\sigma'[\mu'}\eta_{\nu']\rho'}\right)$$

by a little reorganization. Of course this is the Lie algebra relation of $\mathfrak{so}\left(p+1,q+1\right)$ that generate the pseudo-orthogonal group $\mathbb{SO}\left(p+1,q+1\right)$.

Proof:

Of course we have the subgroup of $\mathbb{SO}\left(p+1,q\right)$ generated by the M's which gives us the bulk of $\mathbb{SO}\left(p+1,q+1\right)$, but we need the extra n generators that gives us the full group.

We notice that:

$$2\hat{x}^{\nu}\hat{M}_{\nu\mu}=2\hat{x}^{\nu}\hat{x}_{[\nu}\hat{p}_{\mu]}=2\left(\frac{1}{C}\hat{p}_{\mu}+A\hat{x}_{\mu}\right)$$

$$\implies\hat{p}_{\mu}=C\hat{x}^{\nu}\hat{M}_{\nu\mu}+CA\hat{x}_{\mu}$$

We think of p_{μ} as a function of x and M and the Casimir invariant $\hat{x}\cdot\hat{p}$.

Lets make the convention that the primed indices run from $1,\ldots,n+2$ so we define the extra n generators for $\mathbb{SO}\left(p+1,q+1\right)$ by:

$$\hat{M}_{(n+2)\mu'}=-\hat{M}_{\mu'(n+2)}=\frac{1}{2\sqrt{|C|}}\hat{p}_{\mu'}=\frac{1}{2\sqrt{|C|}}\left(C\hat{x}^{\nu}\hat{M}_{\nu\mu'}+\frac{CA}{i\hbar}\hat{x}_{\mu'}\right)\text{ for }\mu'=1,\ldots,n+1$$

$$\hat{M}_{(n+2)(n+2)}=0$$

and the extra components of η:

$$\eta_{(n+2)(n+2)}=-C/\left|C\right|$$

$$\eta_{(n+2)\mu'}=0\ \text{ for }\mu'\neq n+2$$

We then compute the commutation relation:

$$\left[\hat{M}_{\mu'\nu'}, \hat{M}_{\rho'\sigma'}\right] = i\hbar\left(\hat{M}_{\rho'[\mu'}\eta_{\nu']\sigma'} - \hat{M}_{\sigma'[\mu'}\eta_{\nu']\rho'}\right)$$

Which is equivalent to the commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = 0$$

$$\left[\hat{x}_\mu, \hat{M}_{\nu\rho}\right] = i\hbar\hat{x}_{[\nu}\hat{g}_{\rho]\mu}$$

$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = i\hbar\left(\hat{M}_{\rho[\mu}\eta_{\nu]\sigma} - \hat{M}_{\sigma[\mu}\eta_{\nu]\rho}\right)$$

QED.

We may invert the relations by the formula for x:

$$\hat{x}_\mu = \left(2\hat{M}_{\nu\mu}\hat{p}^\nu + (A + i\hbar)\,\hat{p}_\mu\right)\frac{1}{(\hat{p}^2 - i\hbar CA)}$$

$$\hat{M}_{(n+2)\mu'} = -\hat{M}_{\mu'(n+2)} = \frac{1}{2\sqrt{|C|}}\hat{p}_{\mu'} = \frac{1}{2\sqrt{|C|}}\left(C\hat{x}^\nu\hat{M}_{\nu\mu'} + \frac{CA}{i\hbar}x_{\mu'}\right) \text{ for } \mu' = 1, \ldots, n+1$$

Also, inverting the formulas (6.19) and (6.20) to get $\hat{y}$ in terms of $\hat{x}$ and $\hat{p}$:

$$s^\mu\left(x, \hat{x}\right) = \frac{\hat{x}^\mu}{C\left(\hat{x}^\nu x_\nu\right)} - x^\mu$$

$$k_\mu\left(x, p, \hat{x}, \hat{p}\right) = C\left(\hat{x}^\nu x_\nu\right)\left(\left(\hat{p}_\mu + iC\hbar n\hat{x}_\mu\right) - C\left(\left(\hat{p}_\rho + iC\hbar n\hat{x}_\rho\right)x^\rho\right)x_\mu\right) - p_\mu$$

$$\hat{y}^A\left(x, p, \hat{x}, \hat{p}\right) = \left(s^\mu\left(x, \hat{x}\right), k_\mu\left(x, p, \hat{x}, \hat{p}\right)\right)$$

where $\hat{x}$ and $\hat{p}$ are Hilbert space operators in the observable algebra. Thus we have the consistent interpretation of $\hat{y} = (s, k)$ as an element of some Hilbert space with coefficients which are functions of (x, p).

APPENDIX D

FORMULAS FOR GENERAL PHASE-SPACES

D.1 PROOF OF (5.14) AND (5.15)

Here is the proof that equation (5.11) (also below in (D.1)) imply (5.14) and (5.15).

$$\Omega - D\hat{Q} + \hat{Q}^2/(i\hbar) = 0 \tag{D.1}$$

Before we begin we state a couple useful identities:

$$Df(x,s) = \frac{\partial f}{\partial x^\mu}dx^\mu + \frac{\partial f}{\partial s^\mu}Ds^\mu = dx^\mu\left(\partial_\mu f - \Gamma^\rho{}_{\nu\mu}s^\nu\hat{\partial}_\rho f\right)$$

$$D^2 f(x,s) = dx^\mu dx^\sigma R^\rho{}_{\nu\mu\sigma}s^\nu\hat{\partial}_\rho f$$

for any matrix-valued function $f(x,s)$ where $\hat{\partial}_\mu := \partial/\partial s^\mu$.

Proof:

$$\begin{aligned}
D^2 f(x,s) &= -dx^\mu D\left(\partial_\mu f - \Gamma^\rho{}_{\nu\mu}s^\nu\hat{\partial}_\rho f\right) = -dx^\mu dx^\sigma \nabla_\sigma\left(\partial_\mu f - \Gamma^\rho{}_{\nu\mu}s^\nu\hat{\partial}_\rho f\right) \\
&= -dx^\mu dx^\sigma\left(\partial_\sigma\left(\partial_\mu f - \Gamma^\rho{}_{\nu\mu}s^\nu\hat{\partial}_\rho f\right) - \Gamma^\kappa{}_{\psi\sigma}s^\psi\hat{\partial}_\kappa\left(\partial_\mu f - \Gamma^\rho{}_{\nu\mu}s^\nu\hat{\partial}_\rho f\right)\right) \\
&= dx^\mu dx^\sigma\left(\partial_\sigma\left(\Gamma^\rho{}_{\nu\mu}s^\nu\right)\hat{\partial}_\rho f - \Gamma^\kappa{}_{\psi\sigma}\Gamma^\rho{}_{\nu\mu}s^\psi\hat{\partial}_\kappa\left(s^\nu\hat{\partial}_\rho f\right)\right) \\
&= dx^\mu dx^\sigma\left(-\partial_{[\mu}\Gamma^\rho{}_{\sigma]\nu} + \Gamma^\kappa{}_{\nu[\mu}\Gamma^\rho{}_{\sigma]\kappa}\right)s^\nu\hat{\partial}_\rho f = dx^\mu dx^\sigma R^\rho{}_{\nu\mu\sigma}s^\nu\hat{\partial}_\rho f
\end{aligned}$$

QED.

We make the ansatz:

$$\hat{Q} = \left(k_\nu f^\nu{}_\mu(x,s) + p_\nu g^\nu{}_\mu(x,s) + h_\mu(x,s)\right)dx^\mu + j^\mu(x,s)\,\alpha_\mu \tag{D.2}$$

The components of the tensors $f^{\nu}{}_{\mu}(x,s)$, $g^{\nu}{}_{\mu}(x,s)$, $h_{\mu}(x,s)$, and $j^{\mu}(x,s)$ are matrix-valued functions.

We have from (5.13):

$$\begin{aligned} D\hat{Q} &= \frac{4}{3}R^{\psi}{}_{(\nu\sigma)\beta}dx^{\beta}s^{\sigma}p_{\psi}f^{\nu}{}_{\mu}dx^{\mu} - k_{\nu}dx^{\mu}\left(\Gamma^{\nu}{}_{\rho\sigma}dx^{\sigma}f^{\rho}{}_{\mu} + k_{\nu}Df^{\nu}{}_{\mu}\right) \\ &\quad + D\left(p_{\nu}g^{\nu}{}_{\mu}dx^{\mu} + h_{\mu}dx^{\mu} + j^{\mu}\alpha_{\mu}\right) \end{aligned}$$

$$Dk_{\nu} := -\frac{4}{3}R^{\psi}{}_{(\nu\sigma)\beta}dx^{\beta}s^{\sigma}p_{\psi} + \Gamma^{\rho}{}_{\nu\sigma}dx^{\sigma}k_{\rho}$$

$$\begin{aligned} \hat{Q}^{2}/(i\hbar) &= \left(\left(k_{\nu}f^{\nu}{}_{\mu} + p_{\nu}g^{\nu}{}_{\mu} + h_{\mu}\right)dx^{\mu} + j^{\mu}\alpha_{\mu}\right) \\ &\quad \times \left(\left(k_{\sigma}f^{\sigma}{}_{\rho} + p_{\sigma}g^{\sigma}{}_{\rho} + h_{\rho}\right)dx^{\rho} + j^{\rho}\alpha_{\rho}\right)/(i\hbar) \\ &= \frac{1}{2i\hbar}\left[\left(k_{\nu}f^{\nu}{}_{\mu} + p_{\nu}g^{\nu}{}_{\mu} + h_{\mu}\right), \left(k_{\sigma}f^{\sigma}{}_{\rho} + p_{\sigma}g^{\sigma}{}_{\rho} + h_{\rho}\right)\right]dx^{\mu}dx^{\rho} \\ &\quad + \left[j^{\mu}, k_{\nu}\right]f^{\nu}{}_{\rho}\alpha_{\mu}dx^{\rho}/(i\hbar) \\ &= \frac{1}{2}\begin{pmatrix} \left[k_{\nu}f^{\nu}{}_{\mu} + p_{\nu}g^{\nu}{}_{\mu} + h_{\mu}, k_{\sigma}f^{\sigma}{}_{\rho}\right]/(i\hbar) \\ -f^{\nu}{}_{\mu}\hat{\partial}_{\nu}\left(p_{\sigma}g^{\sigma}{}_{\rho} + h_{\rho}\right) \end{pmatrix} dx^{\mu}dx^{\rho} + \left[j^{\mu}, k_{\nu}\right]f^{\nu}{}_{\rho}\alpha_{\mu}dx^{\rho}/(i\hbar) \\ &= \frac{1}{2}\begin{pmatrix} -2k_{\sigma}f^{\nu}{}_{\mu}\left(\hat{\partial}_{\nu}f^{\sigma}{}_{\rho}\right) + f^{\sigma}{}_{\rho}\left(p_{\nu}\hat{\partial}_{\sigma}g^{\nu}{}_{\mu} + \hat{\partial}_{\sigma}h_{\mu}\right) \\ -f^{\nu}{}_{\mu}\left(p_{\sigma}\hat{\partial}_{\nu}g^{\sigma}{}_{\rho} + \hat{\partial}_{\nu}h_{\rho}\right) \end{pmatrix} dx^{\mu}dx^{\rho} + f^{\nu}{}_{\rho}\left(\hat{\partial}_{\nu}j^{\mu}\right)\alpha_{\mu}dx^{\rho} \end{aligned}$$

$$\hat{Q}^{2} = \left(k_{\nu}f^{\sigma}{}_{\rho}\left(\hat{\partial}_{\sigma}f^{\nu}{}_{\mu}\right) + f^{\sigma}{}_{\rho}\left(p_{\nu}\hat{\partial}_{\sigma}g^{\nu}{}_{\mu} + \hat{\partial}_{\sigma}h_{\mu}\right)\right)dx^{\mu}dx^{\rho} + f^{\nu}{}_{\rho}\left(\hat{\partial}_{\nu}j^{\mu}\right)\alpha_{\mu}dx^{\rho}$$

Plugging these into (D.1):

$$\begin{aligned} 0 &= -R^{\nu}{}_{\mu\sigma\beta}dx^{\sigma}dx^{\beta}k_{\nu}s^{\mu} + \frac{2}{3}D\left(R^{\nu}{}_{(\mu\beta)\sigma}p_{\nu}s^{\beta}s^{\mu}dx^{\sigma}\right) \\ &\quad -\frac{4}{3}R^{\psi}{}_{(\nu\sigma)\beta}dx^{\beta}s^{\sigma}p_{\psi}f^{\nu}{}_{\mu}dx^{\mu} + k_{\nu}dx^{\mu}\left(\Gamma^{\nu}{}_{\rho\sigma}dx^{\sigma}f^{\rho}{}_{\mu} + k_{\nu}Df^{\nu}{}_{\mu}\right) \\ &\quad -D\left(p_{\nu}g^{\nu}{}_{\mu}dx^{\mu} + h_{\mu}dx^{\mu} + j^{\mu}\alpha_{\mu}\right) \\ &\quad + \left(k_{\nu}f^{\sigma}{}_{\rho}\left(\hat{\partial}_{\sigma}f^{\nu}{}_{\mu}\right) + f^{\sigma}{}_{\rho}\left(p_{\nu}\hat{\partial}_{\sigma}g^{\nu}{}_{\mu} + \hat{\partial}_{\sigma}h_{\mu}\right)\right)dx^{\mu}dx^{\rho} + f^{\nu}{}_{\rho}\left(\hat{\partial}_{\nu}j^{\mu}\right)\alpha_{\mu}dx^{\rho} \end{aligned}$$

$$\begin{aligned} 0 &= k_{\nu}dx^{\sigma}\left(R^{\nu}{}_{\mu\beta\sigma}dx^{\beta}s^{\mu} + \left(D + f^{\sigma}{}_{\rho}dx^{\rho}\hat{\partial}_{\sigma}\right)f^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\rho\mu}dx^{\mu}f^{\rho}{}_{\sigma}\right) \\ &\quad + \frac{2}{3}D\left(R^{\nu}{}_{(\mu\beta)\sigma}p_{\nu}s^{\beta}s^{\mu}dx^{\sigma}\right) - \frac{4}{3}R^{\psi}{}_{(\nu\sigma)\beta}dx^{\beta}s^{\sigma}p_{\psi}f^{\nu}{}_{\mu}dx^{\mu} \\ &\quad - \left(D + f^{\sigma}{}_{\rho}dx^{\rho}\hat{\partial}_{\sigma}\right)\left(p_{\nu}g^{\nu}{}_{\mu}dx^{\mu} + h_{\mu}dx^{\mu} + j^{\mu}\alpha_{\mu}\right) \end{aligned}$$

Let P be the differential operator:

$$P := D + f^{\sigma}{}_{\rho} dx^{\rho} \hat{\partial}_{\sigma}$$

then the above condition simplifies in appearance to:

$$\begin{aligned} 0 &= k_{\nu} dx^{\sigma} \left(R^{\nu}{}_{\mu\beta\sigma} dx^{\beta} s^{\mu} + \left(D + f^{\sigma}{}_{\rho} dx^{\rho} \hat{\partial}_{\sigma} \right) f^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\rho\mu} dx^{\mu} f^{\rho}{}_{\sigma} \right) \\ & \underbrace{+\frac{2}{3} D \left(R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} s^{\mu} dx^{\sigma} \right) - \frac{4}{3} R^{\psi}{}_{(\nu\sigma)\beta} dx^{\beta} s^{\sigma} p_{\psi} f^{\nu}{}_{\mu} dx^{\mu}}_{(A)} \\ & - \left(D + f^{\sigma}{}_{\rho} dx^{\rho} \hat{\partial}_{\sigma} \right) \left(p_{\nu} g^{\nu}{}_{\mu} dx^{\mu} + h_{\mu} dx^{\mu} + j^{\mu} \alpha_{\mu} \right) \end{aligned} \tag{D.3}$$

$$\begin{aligned} (A) &= \frac{2}{3} D \left(R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} s^{\mu} dx^{\sigma} \right) - \frac{4}{3} R^{\psi}{}_{(\nu\sigma)\beta} dx^{\beta} s^{\sigma} p_{\psi} f^{\nu}{}_{\mu} dx^{\mu} \\ &= \frac{2}{3} P \left(R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} s^{\mu} dx^{\sigma} \right) - \frac{2}{3} f^{\psi}{}_{\rho} dx^{\rho} \hat{\partial}_{\psi} \left(R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} s^{\mu} dx^{\sigma} \right) \\ & -\frac{4}{3} R^{\psi}{}_{(\nu\sigma)\beta} s^{\sigma} p_{\psi} f^{\nu}{}_{\mu} dx^{\beta} dx^{\mu} \\ &= \frac{2}{3} P \left(R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} s^{\mu} dx^{\sigma} \right) - \frac{4}{3} R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} f^{\mu}{}_{\rho} dx^{\rho} dx^{\sigma} \\ & -\frac{4}{3} R^{\nu}{}_{(\mu\beta)\sigma} s^{\beta} p_{\nu} f^{\mu}{}_{\rho} dx^{\sigma} dx^{\rho} \\ &= \frac{2}{3} P \left(R^{\nu}{}_{(\mu\beta)\sigma} p_{\nu} s^{\beta} s^{\mu} dx^{\sigma} \right) \end{aligned}$$

Putting (A) back into the condition at (D.3):

$$\begin{aligned} 0 &= k_{\nu} dx^{\sigma} \left(R^{\nu}{}_{\mu\beta\sigma} dx^{\beta} s^{\mu} + P f^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\rho\mu} dx^{\mu} f^{\rho}{}_{\sigma} \right) \\ & -P \left(p_{\nu} \left(g^{\nu}{}_{\mu} dx^{\mu} - \frac{2}{3} R^{\nu}{}_{(\mu\beta)\sigma} s^{\beta} s^{\mu} dx^{\sigma} \right) + h_{\mu} dx^{\mu} + j^{\mu} \alpha_{\mu} \right) \end{aligned} \tag{D.4}$$

Let:

$$\boxed{g^{\nu}{}_{\sigma} = w^{\nu}{}_{\sigma} - \tfrac{2}{3} R^{\nu}{}_{(\mu\beta)\sigma} s^{\beta} s^{\mu}} \tag{D.5}$$

Putting this into (D.4):

$$\begin{aligned} 0 &= k_{\nu} dx^{\sigma} \left(R^{\nu}{}_{\mu\beta\sigma} dx^{\beta} s^{\mu} + P f^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\rho\mu} dx^{\mu} f^{\rho}{}_{\sigma} \right) \\ & \underbrace{-P \left(p_{\nu} w^{\nu}{}_{\mu} dx^{\mu} + j^{\mu} \alpha_{\mu} \right)}_{(B)} - P \left(h_{\mu} dx^{\mu} \right) \end{aligned} \tag{D.6}$$

$$(B) = -\left(P p_{\nu} \right) w^{\nu}{}_{\sigma} dx^{\sigma} - p_{\nu} P \left(w^{\nu}{}_{\sigma} dx^{\sigma} \right) - \left(P j^{\mu} \right) \alpha_{\mu} - j^{\mu} \left(P \alpha_{\mu} \right)$$

Computing and simplifying the expressions:

$$P\alpha_\mu = D\alpha_\mu = -\frac{4}{3}R^\nu{}_{(\mu\beta)\sigma}p_\nu dx^\beta dx^\sigma + \Gamma^\nu{}_{\mu\sigma}dx^\sigma\alpha_\nu$$

$$Pp_\nu = Dp_\nu = dp_\nu = \alpha_\nu + \Gamma^\gamma{}_{\nu\rho}dx^\rho p_\gamma$$

$$-\frac{4}{3}R^\nu{}_{(\mu\beta)\sigma}p_\nu j^\mu dx^\beta dx^\sigma = \frac{2}{3}\left(-R^\nu{}_{(\mu\beta)\sigma} + R^\nu{}_{(\mu\sigma)\beta}\right)p_\nu j^\mu dx^\beta dx^\sigma = -R^\nu{}_{\mu\beta\sigma}p_\nu j^\mu dx^\beta dx^\sigma$$

we get:

$$(B) = \alpha_\mu\left(Pj^\mu + \Gamma^\mu{}_{\nu\sigma}dx^\sigma j^\nu - w^\mu{}_\sigma dx^\sigma\right) + p_\nu dx^\sigma\left(Pw^\nu{}_\sigma + \Gamma^\nu{}_{\kappa\rho}dx^\rho w^\kappa{}_\sigma\right) + R^\nu{}_{\mu\beta\sigma}p_\nu j^\mu dx^\beta dx^\sigma$$

and let:

$$\boxed{w^\nu{}_\sigma dx^\sigma = Pj^\nu + \Gamma^\nu{}_{\rho\sigma}dx^\sigma j^\rho} \tag{D.7}$$

then:

$$\begin{aligned}(B) &= p_\nu\left(P^2 j^\nu + P\left(\Gamma^\nu{}_{\rho\sigma}dx^\sigma\right)j^\rho - \Gamma^\nu{}_{\rho\sigma}dx^\sigma\left(Pj^\rho\right) + \Gamma^\nu{}_{\kappa\rho}dx^\rho\left(Pj^\kappa + \Gamma^\kappa{}_{\gamma\beta}dx^\beta j^\gamma\right)\right) \\ &\quad + R^\nu{}_{\mu\beta\sigma}p_\nu j^\mu dx^\beta dx^\sigma \\ &= p_\nu\left(P^2 j^\nu + \left(\partial_{[\mu}\Gamma^\nu{}_{\sigma]\rho} - \Gamma^\kappa{}_{\rho[\mu}\Gamma^\nu{}_{\sigma]\kappa}\right)dx^\mu dx^\sigma j^\rho\right) + R^\nu{}_{\mu\beta\sigma}p_\nu j^\mu dx^\beta dx^\sigma\end{aligned}$$

$$(B) = p_\nu P^2 j^\nu \tag{D.8}$$

We have the identity:

$$P^2 = D^2 + \frac{4}{3}p_\delta R^\delta{}_{(\beta\kappa)\gamma}f^\beta{}_\rho dx^\kappa dx^\rho\left[s^\gamma,\cdot\right] + \left(Pf^\sigma{}_\rho + \Gamma^\sigma{}_{\beta\kappa}dx^\kappa f^\beta{}_\rho\right)dx^\rho\hat{\partial}_\sigma \tag{D.9}$$

Proof of (D.9)**:**

$$\begin{aligned}P^2 h &= \left(D + f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma\right)^2 h \\ &= \left(D + f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma\right)\left(D + f^\psi{}_\kappa dx^\kappa\hat{\partial}_\psi\right)h \\ &= D^2 h + \underbrace{f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma\left(Dh\right) + D\left(f^\psi{}_\kappa dx^\kappa\hat{\partial}_\psi h\right)}_{(C)} + \underbrace{f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma\left(f^\psi{}_\kappa dx^\kappa\hat{\partial}_\psi h\right)}_{(E)}\end{aligned} \tag{D.10}$$

We know that:

$$\hat{\partial}_\psi h = \left[k_\psi, h\right]/\left(i\hbar\right)$$

$$Dk_\sigma := -\frac{4}{3}R^\delta{}_{(\sigma\kappa)\beta}dx^\kappa s^\beta p_\delta + \Gamma^\rho{}_{\sigma\kappa}dx^\kappa k_\rho$$

We can easily prove that $D([f,h]) - [f,Dh] = [Df,h]$ for any matrix-valued functions $f(x,p,s,k)$ and $h(x,p,s,k)$ therefore:

$$i\hbar D\left(\hat{\partial}_\psi h\right) - i\hbar\hat{\partial}_\psi(Dh) = i\hbar\left[-\frac{4}{3}R^\delta{}_{(\psi\sigma)\beta}dx^\sigma s^\beta p_\delta + \Gamma^\rho{}_{\psi\sigma}dx^\sigma k_\rho, h\right]$$

and (C) becomes:

$$\begin{aligned}
(C) &= i\hbar\left(f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma(Dh) + D\left(f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma h\right)\right) \\
&= f^\sigma{}_\rho dx^\rho\left([k_\sigma, Dh] - D([k_\sigma,h])\right) + D\left(f^\sigma{}_\rho dx^\rho\right)[k_\sigma,h] \\
&= f^\sigma{}_\rho dx^\rho[Dk_\sigma,h] + D\left(f^\sigma{}_\rho dx^\rho\right)[k_\sigma,h] \\
&= f^\psi{}_\rho dx^\rho\left(-\frac{4}{3}R^\delta{}_{(\psi\kappa)\beta}dx^\kappa\left[s^\beta,h\right]p_\delta + \Gamma^\sigma{}_{\psi\kappa}dx^\kappa[k_\sigma,h]\right) + \left(Df^\sigma{}_\rho\right)dx^\rho[k_\sigma,h]
\end{aligned}$$

$$\implies (C) = \frac{4}{3}p_\delta R^\delta{}_{(\psi\kappa)\beta}f^\psi{}_\rho dx^\kappa dx^\rho\left[s^\beta,h\right] + \left(Df^\sigma{}_\rho + f^\psi{}_\rho\Gamma^\sigma{}_{\psi\kappa}dx^\kappa\right)dx^\rho\hat{\partial}_\sigma h$$

also:

$$\begin{aligned}
(E) &= f^\sigma{}_\rho dx^\rho dx^\kappa\left(\hat{\partial}_\sigma f^\psi{}_\kappa\right)\hat{\partial}_\psi h + f^\sigma{}_\rho dx^\rho f^\psi{}_\kappa dx^\kappa\left(\hat{\partial}_\sigma\hat{\partial}_\psi h\right) \\
&= f^\sigma{}_\rho dx^\rho dx^\kappa\left(\hat{\partial}_\sigma f^\psi{}_\kappa\right)\hat{\partial}_\psi h
\end{aligned}$$

Putting (C) and (E) into the condition at (D.10):

$$P^2h = D^2h + \frac{4}{3}p_\delta R^\delta{}_{(\psi\kappa)\beta}f^\psi{}_\rho dx^\kappa dx^\rho\left[s^\beta,h\right] + \left(Pf^\sigma{}_\rho + f^\psi{}_\rho dx^\kappa\Gamma^\sigma{}_{\psi\kappa}\right)dx^\rho\hat{\partial}_\sigma h$$

$$\implies P^2 = D^2 + \frac{4}{3}p_\delta R^\delta{}_{(\beta\kappa)\gamma}f^\beta{}_\rho dx^\kappa dx^\rho[s^\gamma,\cdot] + \left(Pf^\sigma{}_\rho + \Gamma^\sigma{}_{\beta\kappa}dx^\kappa f^\beta{}_\rho\right)dx^\rho\hat{\partial}_\sigma$$

QED.

Using this identity (D.9) in (B) at equation (D.8):

$$\begin{aligned}
(B) &= p_\nu\left(D^2j^\nu + \left(Pf^\sigma{}_\rho + \Gamma^\sigma{}_{\beta\kappa}dx^\kappa f^\beta{}_\rho\right)dx^\rho\hat{\partial}_\sigma j^\nu\right) \\
&= \left(Pf^\sigma{}_\rho + \Gamma^\sigma{}_{\beta\kappa}dx^\kappa f^\beta{}_\rho - dx^\kappa R^\sigma{}_{\beta\rho\kappa}s^\beta\right)p_\nu dx^\rho\hat{\partial}_\sigma j^\nu
\end{aligned}$$

Putting (B) back into (D.6):

$$\begin{aligned} 0 &= k_\nu dx^\sigma \left(Pf^\nu{}_\sigma + \Gamma^\nu{}_{\rho\mu} dx^\mu f^\rho{}_\sigma + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta\right) \\ &\quad \left(Pf^\sigma{}_\rho + \Gamma^\sigma{}_{\beta\kappa} dx^\kappa f^\beta{}_\rho + R^\sigma{}_{\beta\kappa\rho} s^\beta dx^\kappa\right) p_\nu dx^\rho \hat{\partial}_\sigma j^\nu - P\left(h_\mu dx^\mu\right) \end{aligned} \tag{D.11}$$

If $h_\mu dx^\mu = 0$, then we see that the only condition for (D.2) to be a solution to (5.11) is:

$$\left(Pf^\nu{}_\sigma + \Gamma^\nu{}_{\rho\mu} dx^\mu f^\rho{}_\sigma + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta\right) dx^\sigma = 0$$

or:

$$\left(\left(D + f^\mu{}_\rho dx^\rho \hat{\partial}_\mu\right) f^\nu{}_\sigma + \Gamma^\nu{}_{\rho\mu} dx^\mu f^\rho{}_\sigma + R^\nu{}_{\mu\beta\sigma} s^\mu dx^\beta\right) dx^\sigma = 0$$

which is precisely equation (5.15).

Putting (D.5) and (D.7) into what we started out with in equation (D.2):

$$\begin{aligned} g^\nu{}_\mu &= Pj^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu \\ &= \left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma\right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu \end{aligned}$$

$$P := D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma$$

into :

$$\hat{Q} = j^\mu \alpha_\mu + k_\nu f^\nu{}_\mu dx^\mu + p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma\right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma\right)$$

We redefine $j^\mu \to j^\mu + s^\mu$ and $f^\nu{}_\mu \to f^\nu{}_\mu - \delta^\nu_\mu$:

$$\begin{aligned} \hat{Q} &= \left(s^\mu \alpha_\mu - k_\mu dx^\mu\right) + j^\mu \alpha_\mu + k_\nu f^\nu{}_\mu dx^\mu \\ &\quad + p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - dx^\sigma \hat{\partial}_\sigma\right)\left(j^\nu + s^\nu\right) + \Gamma^\nu{}_{\rho\sigma} dx^\sigma \left(j^\rho + s^\rho\right) - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma\right) \end{aligned}$$

$$\begin{aligned} &= \left(s^\mu \alpha_\mu - k_\mu dx^\mu\right) + j^\mu \alpha_\mu + k_\nu f^\nu{}_\mu dx^\mu + p_\nu \left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - dx^\sigma \hat{\partial}_\sigma\right) s^\nu + p_\nu \Gamma^\nu{}_{\rho\sigma} dx^\sigma s^\rho \\ &\quad + p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - dx^\sigma \hat{\partial}_\sigma\right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma\right) \end{aligned}$$

$$\begin{aligned} \hat{Q} &= \left(s^\mu \alpha_\mu - z_\mu dx^\mu\right) + j^\mu \alpha_\mu + z_\nu f^\nu{}_\mu dx^\mu \\ &\quad + p_\nu \left(\left(D + f^\sigma{}_\rho dx^\rho \hat{\partial}_\sigma - dx^\sigma \hat{\partial}_\sigma\right) j^\nu + \Gamma^\nu{}_{\rho\sigma} dx^\sigma j^\rho - \frac{2}{3} R^\nu{}_{(\mu\beta)\sigma} s^\beta s^\mu dx^\sigma\right) \end{aligned} \tag{D.12}$$

which is precisely the ansatz in (5.14).

To see that the term:

$$p_\nu\left(\left(D+f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma-dx^\sigma\hat{\partial}_\sigma\right)j^\nu+\Gamma^\nu{}_{\rho\sigma}dx^\sigma j^\rho\right)$$

in (5.14) is coordinate-free object we express it in terms of abstract indices:

$$p_b\left(\nabla_c j^b+f^e{}_c\hat{\partial}_e-\hat{\partial}_c\right)j^b$$

where we used the fact that $Dj^b=\Theta^C D_C j^b=\nabla_c j^b$ because j^b is a function of x and s only.

So we can see that if j^b and $f^e{}_c$ are tensors on the configuration space then $\hat{Q}$ is coordinate-free.

The condition on $f^\nu{}_\mu$ is:

$$\left(\left(D+f^\mu{}_\rho dx^\rho\hat{\partial}_\mu-dx^\mu\hat{\partial}_\mu\right)f^\nu{}_\sigma+\Gamma^\nu{}_{\rho\mu}dx^\mu f^\rho{}_\sigma-\Gamma^\nu{}_{\sigma\mu}dx^\mu+R^\nu{}_{\mu\beta\sigma}s^\mu dx^\beta\right)dx^\sigma=0 \qquad \text{(D.13)}$$

which is the same as (5.15).

D.2 THE PROOF OF THE INTEGRABILITY OF (5.15)

We want to show that the condition (5.15) is integrable locally. Showing that the following the P-derivative of the condition in (5.15) vanishes:

$$P\,(5.15)=P\left(\left(P-dx^\mu\hat{\partial}_\mu\right)f^\nu{}_\sigma+\Gamma^\nu{}_{\rho\mu}dx^\mu f^\rho{}_\sigma-\Gamma^\nu{}_{\sigma\mu}dx^\mu+R^\nu{}_{\mu\beta\sigma}s^\mu dx^\beta\right)dx^\sigma=0 \qquad \text{(D.14)}$$

where P is the differential operator:

$$P=\left(D+f^\sigma{}_\rho dx^\rho\hat{\partial}_\sigma\right)$$

where $\hat{\partial}_\mu:=\partial/\partial s^\mu$ implies that the condition (5.15) is integrable locally by the Cauchy-Kovalevskaya theorem.

*Note this analogous to how Fedosov can locally integrate the solution for $\hat{D}$, i.e., by requiring that $\left(D-\hat{D}\right)^2\hat{y}^A=0$ in (3.8). However, before doing this by brute force we notice that D acting on everything in the equation above is just the configuration space connection ∇.

Therefore, to simplify the calculation we will us abstract indices. The equation above in (D.14) (and in (5.15)) becomes the equation:

$$\left(\nabla_{[n} + f^{d}_{\ [n}\hat{\partial}_d\right)\left(R_{ca]\ \ m}^{\ \ \ \ b} s^m + \left(\nabla_c + f^{e}_{\ c}\hat{\partial}_{|e|}\right) f^{b}_{\ a]}\right) = 0 \tag{D.15}$$

where P on configuration space quantities is $\left(\nabla_n + f^{d}_{\ n}\hat{\partial}_d\right)$.

First we note that we want f^a_b to be a globally defined object hence it should be made out of tensors. This rules out the trivial solution of $f^{\sigma}_{\ \rho} dx^{\rho} = -\Gamma^{\sigma}_{\ \rho\nu} s^{\nu} dx^{\rho}$.

Proof:

$$\begin{aligned}
(\text{D.15}) &= \left(\nabla_{[n} + f^{d}_{\ [n}\hat{\partial}_d\right)\left(R_{ca]\ \ m}^{\ \ \ \ b} s^m + \left(\nabla_c + f^{e}_{\ c}\hat{\partial}_{|e|}\right) f^{b}_{\ a]}\right) \\
&= \left(\nabla_{[n} R_{ca]\ \ m}^{\ \ \ \ b}\right) s^m - R^{b}_{\ m[ac}\left(\nabla_{n]} + f^{d}_{\ n]}\hat{\partial}_d\right) s^m + \left(\nabla_{[n} + f^{d}_{\ [n}\hat{\partial}_d\right)\left(\nabla_c + f^{e}_{\ c}\hat{\partial}_{|e|}\right) f^{b}_{\ a]}
\end{aligned}$$

In abstract indices we have the identities:

$$Ds^a = 0$$

$$\nabla_a f(x,s) = \partial_a f - \Gamma^{c}_{\ ab} s^b \hat{\partial}_c f$$

$$\nabla_{[n}\nabla_{c]} f(x,s) = R^{b}_{\ enc} s^e \hat{\partial}_b f$$

We also have the second Bianchi identity:

$$\nabla_{[n} R_{ca]\ \ m}^{\ \ \ \ b} = 0$$

The identity in (D.9):

$$P^2 = D^2 + \frac{4}{3} p_{\delta} R^{\delta}_{\ (\beta\kappa)\gamma} f^{\beta}_{\ \rho} dx^{\kappa} dx^{\rho}\left[s^{\gamma}, \cdot\right] + \left(P f^{\sigma}_{\ \rho} + \Gamma^{\sigma}_{\ \beta\kappa} dx^{\kappa} f^{\beta}_{\ \rho}\right) dx^{\rho}\hat{\partial}_{\sigma}$$

in abstract indices is:

$$\begin{aligned}
&\left(\nabla_{[n} + f^{d}_{\ [n}\hat{\partial}_d\right)\left(\nabla_{c]} + f^{e}_{\ c]}\hat{\partial}_e\right) h \\
= \ & \nabla_{[n}\nabla_{c]} h + \frac{2}{3} p_d\left(R^{d}_{\ (mc)a} f^{m}_{\ n} - R^{d}_{\ (mn)a} f^{m}_{\ c}\right)\left[s^a, h\right] + \left(\left(\nabla_{[n} + f^{d}_{\ [n}\hat{\partial}_{|d|}\right) f^{e}_{\ c]}\right)\hat{\partial}_e h \\
= \ & \nabla_{[n}\nabla_{c]} h + \frac{2}{3} p_d\left(R^{d}_{\ (mc)a} f^{m}_{\ n} - R^{d}_{\ (mn)a} f^{m}_{\ c}\right)\left[s^a, h\right] + \left(\left(\nabla_{[n} f^{e}_{\ c]} + \left(\hat{\partial}_d f^{e}_{\ [c}\right) f^{d}_{\ n]}\right)\right)\hat{\partial}_e h
\end{aligned}$$

where h is an arbitrary matrix-valued function of x and s.

Condition (D.15) becomes:

$$\begin{aligned}
(D.15) &= -R^{b}{}_{m[ac}f^{m}{}_{n]} + \nabla_{[n}\nabla_{c}f^{b}{}_{a]} + \left(\nabla_{[n}f^{e}{}_{c} + \left(\hat{\partial}_{d}f^{e}{}_{[c}\right)f^{d}{}_{n}\right)\hat{\partial}_{|e|}f^{b}{}_{a]} \\
&= -R^{b}{}_{m[ac}f^{m}{}_{n]} + R^{d}{}_{[anc]}f^{b}{}_{d} - R^{b}{}_{d[nc}f^{d}{}_{a]} + \left(\hat{\partial}_{d}f^{b}{}_{[a}\right)R_{nc]}{}^{d}{}_{e}s^{e} \\
&\quad + \left(\nabla_{[n}f^{e}{}_{c} + \left(\hat{\partial}_{d}f^{e}{}_{[c}\right)f^{d}{}_{n}\right)\hat{\partial}_{|e|}f^{b}{}_{a]} \\
&= \left(\hat{\partial}_{d}f^{b}{}_{[a}\right)R_{nc]}{}^{d}{}_{e}s^{e} + \left(\nabla_{[n}f^{d}{}_{c} + \left(\hat{\partial}_{e}f^{d}{}_{[c}\right)f^{e}{}_{n}\right)\hat{\partial}_{|d|}f^{b}{}_{a]} \\
&= \hat{\partial}_{d}f^{b}{}_{[a}\left(R_{nc]}{}^{d}{}_{e}s^{e} + \nabla_{n}f^{d}{}_{c]} + \left(\hat{\partial}_{|e|}f^{d}{}_{c}\right)f^{e}{}_{n]}\right)
\end{aligned}$$

Now according to the original condition (5.15) becomes the equation:

$$\nabla_{[n}f^{d}{}_{c]} + \left(\hat{\partial}_{e}f^{d}{}_{[c}\right)f^{e}{}_{n]} = -R^{d}{}_{enc}s^{e}$$

therefore:

$$(D.15) = \hat{\partial}_{d}f^{b}{}_{[a}\left(R_{nc]}{}^{d}{}_{e}s^{e} - R^{d}{}_{|e|nc]}s^{e}\right) = 0$$

so modulo the original condition (5.15) the local integrability condition (D.15) is zero. Therefore $f^{b}{}_{a}$ exists at least locally by the Cauchy-Kovalevskaya theorem.

QED.

D.3 PROOF THAT (I.7) IS THE SOLUTION TO (I.6)

We want to prove that the solution to equation (I.6):

$$S^{b}{}_{[ac]e} = 0 \quad \& \quad S^{b}{}_{a[ce]} = -R^{b}{}_{ace}$$

is equation (I.7):

$$S^{b}{}_{ace} = -\frac{4}{3}R^{b}{}_{(ac)e}$$

Proof:

Let:

$$S^{b}{}_{ace} = -\frac{4}{3}R^{b}{}_{(ac)e}$$

then the first condition is automatically satisfied:

$$S^b{}_{[ac]e} = 0$$

Now we need to show that:

$$S^b{}_{a[ce]} = -R^b{}_{ace}$$

I.e. we need to show that:

$$-\frac{2}{3}R^b{}_{(ac)e} + \frac{2}{3}R^b{}_{(ae)c} = -R^b{}_{ace}$$

$$\begin{aligned} -\frac{2}{3}R^b{}_{(ac)e} + \frac{2}{3}R^b{}_{(ae)c} &= -\frac{1}{3}\left(R^b{}_{ace} + R^b{}_{cae} - R^b{}_{aec} - R^b{}_{eac}\right) \\ &= -\frac{1}{3}\left(2R^b{}_{ace} + R^b{}_{cae} + \left(R^b{}_{cea} + R^b{}_{ace}\right)\right) = -R^b{}_{ace} \end{aligned}$$

QED.

Related formulas are:

$$\frac{2}{3}\left(R^b{}_{(ac)e} - R^b{}_{(ae)c}\right) = R^b{}_{ace}$$

D.4 SYMMETRIES FORMULAS FOR CURVATURE

The standard symmetries of R_{abcd} are:

$$R_{abcd} = R_{ab[cd]} = R_{[ab]cd} = R_{cdab}$$

The first Bianchi identity:

$$R_{a[bcd]} = 0$$

In alternative and more convenient form:

$$R_{abcd} + R_{adbc} + R_{acdb} = 0$$

A useful identity related to the above identities is:

$$\frac{2}{3}\left(R^b{}_{(ac)e} - R^b{}_{(ae)c}\right) = R^b{}_{ace}$$

and the second Bianchi identity:

$$\nabla_{[a}R_{bc]de} = 0$$

Formula for curvature:

$$R^a{}_{bce} = -\partial_{[c}\Gamma^a{}_{e]b} + \Gamma^h{}_{b[c}\Gamma^a{}_{e]h}$$

$$R^\mu{}_{\nu\sigma\rho} = -\partial_{[\sigma}\Gamma^\mu{}_{\rho]\nu} + \Gamma^\kappa{}_{\nu[\sigma}\Gamma^\mu{}_{\rho]\kappa}$$

APPENDIX E

PHASE-SPACE CURVATURE FOR CONSTANT CURVATURE MANIFOLDS OF CODIMENSION ONE

E.1 CALCULATION OF THE CONFIGURATION SPACE CURVATURE AND CONNECTION FOR THE GENERAL CASE

E.1.1 Proof of The Configuration Space Connection in (6.10)

We want to show that the Configuration Space Connection is (6.10) (also (E.1):

$$\Gamma^{\mu}{}_{\nu\sigma} = Cx^{\mu}g_{\nu\sigma} - 2Cx_{(\nu}\left(\delta^{\mu}_{\sigma)} - Cx_{\sigma)}x^{\mu}\right) \tag{E.1}$$

given the formula

$$\Gamma^{\rho}{}_{\mu\nu} = -\frac{1}{2}g^{\rho\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right) \tag{E.2}$$

$$g_{\mu\nu} = \eta_{\mu\nu} - Cx_{\mu}x_{\nu}$$

$$\nabla_{\sigma}dx^{\mu} = -\Gamma^{\mu}{}_{\nu\sigma}dx^{\nu}$$

Γ must satisfy the following conditions:

1. torsion-free:

$$dx^{\sigma}\nabla_{\sigma}\left(dx^{\mu}\right) = -\Gamma^{\mu}{}_{\nu\sigma}dx^{\sigma}dx^{\nu} \implies \Gamma^{\mu}{}_{[\nu\sigma]} = 0 \tag{E.3}$$

2. metric-preserving:

$$\nabla_\rho \left(g_{\mu\nu} dx^\mu dx^\nu\right) = 0 \tag{E.4}$$

3. The directional derivative $\mathcal{D}_v$ of a vector and covector in any direction v^a is also a vector and covector respectively.

$$w_\mu \text{ is a covector } \iff \mathcal{D}_v w_\mu = v^\rho \left(\partial_\rho w_\mu - \Gamma^\nu{}_{\mu\rho} w_\nu\right) \text{ is a covector} \tag{E.5}$$

$$w^\mu \text{ is a vector } \iff \mathcal{D}_v w^\mu = v^\rho \left(\partial_\rho w^\mu + \Gamma^\mu{}_{\nu\rho} w^\nu\right) \text{ is a vector}$$

4. The constraints:

$$x^\mu x_\mu = 1/C \tag{E.6}$$

$$x^\mu p_\mu = 0 \tag{E.7}$$

$$x^\mu dx_\mu = 0 \tag{E.8}$$

$$p_\mu dx^\mu + x^\mu dp_\mu = 0 \tag{E.9}$$

$$\nabla_\nu \left(x^\mu dx_\mu\right) = 0 \tag{E.10}$$

$$D \otimes \left(p_\mu dx^\mu + x^\mu dp_\mu\right) = 0 \tag{E.11}$$

Proof:

The formula (E.2) is derived from the condition (E.4) and results in the formula for the symbols:

$$\Gamma^\rho{}_{\mu\nu} = \frac{C}{2} g^{\rho\sigma} \left(\partial_\mu \left(x_\nu x_\sigma\right) + \partial_\nu \left(x_\mu x_\sigma\right) - \partial_\sigma \left(x_\mu x_\nu\right)\right) = 0$$

However, remember we have constraints $x^\mu x_\mu = 1/C$ and $x^\mu dx_\mu = 0$ and remember that the above equation comes from preserving the metric i.e.:

$$\nabla_\nu \left(g_{\rho\sigma} dx^\rho dx^\sigma\right) = \left(\partial_\nu g_{\rho\sigma}\right) dx^\rho dx^\sigma - 2 g_{\rho\sigma} \Gamma^\rho{}_{\mu\nu} dx^\mu dx^\sigma$$

we see the replacement ${\Gamma^\rho}_{\mu\nu} \rightarrow {\Gamma^\rho}_{\mu\nu} + x^\rho q_{\mu\nu} + x_{(\mu} {f^\rho}_{\nu)}$ where $q_{(\mu\nu)} = q_{\mu\nu}$ leaves this equation invariant and the connection torsion-free. So we can define ${\Gamma^\rho}_{\mu\nu}$ is defined as:

$$ {\Gamma^\rho}_{\mu\nu} := x^\rho q_{\mu\nu} + 2x_{(\mu} {f^\rho}_{\nu)} \tag{E.12} $$

where $q_{\mu\nu}$ and ${f^\rho}_\nu$ need to be determined by imposing the various constraints.

Lemma: The condition:

$$ x^\mu \alpha_\mu = 0 \tag{E.13} $$

is equivalent to the condition in (E.5).

Proof:

Let $p_\mu = w_\mu(x)$ for any functions w_μ and look at the directional derivative in an arbitrary direction v^μ:

$$ \mathcal{D}_v w_\mu = v^\rho \left(\partial_\rho w_\mu - {\Gamma^\nu}_{\mu\rho} w_\nu \right) $$

From the condition in (E.5) we require that $\mathcal{D}_v w_\mu$ be a covector on the constraint surface $x^\mu x_\mu = 1/C$ at x just as w_μ is a covector is and it must obey the condition below just as w_μ does:

$$ w_\mu x^\mu = 0 \iff \left(\mathcal{D}_v w_\mu \right) x^\mu = 0 $$

Since this must be satisfied for all sections in the cotangent bundle $w_\mu(x)$ and all directions v^μ then by the definition of α_μ:

$$ \alpha_\mu := dp_\mu - {\Gamma^\nu}_{\mu\rho} dx^\rho p_\nu $$

we require:

$$ x^\mu \alpha_\mu = 0 $$

which is equivalent to the constraint in (E.5).

QED.

Putting (5.5) into the constraint (E.13):

$$ 0 = x^\mu \alpha_\mu = x^\mu \left(dp_\mu - {\Gamma^\nu}_{\mu\rho} dx^\rho p_\nu \right) = x^\mu dp_\mu - x^\mu {\Gamma^\nu}_{\mu\rho} dx^\rho p_\nu $$

$$ 0 = x^\mu dp_\mu - x^\mu {\Gamma^\nu}_{\mu\rho} dx^\rho p_\nu $$

we have the constraint in (E.9) that $x^\mu dp_\mu = -p_\mu dx^\mu$:

$$0 = -p_\mu dx^\mu - x^\mu \left(x^\nu q_{\mu\rho} + 2x_{(\mu} f^\nu{}_{\rho)}\right) dx^\rho p_\nu$$

so the obtain a condition for q and f:

$$0 = -p_\mu dx^\mu - A x^\mu q_{\mu\rho} dx^\rho - \frac{1}{C} f^\nu{}_\rho dx^\rho p_\nu \tag{E.14}$$

$\boxed{\text{(E.10)}}$

Now we derive another condition for q and f by putting (5.5) into constraint (E.10) we get:

$$0 = \nabla_\mu \left(x_\rho dx^\rho\right) = dx_\mu - x_\rho \Gamma^\rho{}_{\mu\nu} dx^\nu = dx_\mu - x_\rho \left(x^\rho q_{\mu\nu} + 2x_{(\mu} f^\rho{}_{\nu)}\right) dx^\nu$$

$$\implies q_{\mu\nu} dx^\nu = C dx_\mu + C x_\rho x_\mu f^\rho{}_\nu dx^\nu \tag{E.15}$$

and substituting this into (E.14):

$$0 = -p_\mu dx^\mu - A x_\rho f^\rho{}_\nu dx^\nu - \frac{1}{C} f^\nu{}_\rho dx^\rho p_\nu$$

$$\implies -\left(\delta^\nu_\mu + \frac{1}{C} f^\nu{}_\mu\right) p_\nu - A x_\rho f^\rho{}_\mu \sim x_\mu$$

means that $\left(\delta^\nu_\mu + \frac{1}{C} f^\nu{}_\mu\right)$ must be proportional to x^ν *and* x_μ.

$$\implies \delta^\nu_\mu + \frac{1}{C} f^\nu{}_\mu = -B x^\nu x_\mu$$

$$f^\nu{}_\mu = -C\left(\delta^\nu_\mu - B x^\nu x_\mu\right) \tag{E.16}$$

where B is some arbitrary constant.

Putting this back into (E.15):

$$q_{\mu\nu} dx^\nu = C dx_\mu - C^2 x_\rho x_\mu \left(\delta^\rho_\nu - B x^\rho x_\nu\right) dx^\nu = C dx_\mu$$

$$q_{\mu\nu} = C\left(\eta_{\mu\nu} - F x_\mu x_\nu\right) \tag{E.17}$$

where F is some arbitrary constant and we remember that $q_{(\mu\nu)} = q_{\mu\nu}$.

The choice of constants B and F have no consequence in our formulas for the covariant derivative preserving any of the constraints, the metric, etc. and we can set them as anything we'd like. We choose $F = B = C$ which means that:

$$q_{\mu\nu} = Cg_{\mu\nu} \quad , \quad f^{\nu}{}_{\mu} = -C\left(\delta^{\nu}_{\mu} - Cx^{\nu}x_{\mu}\right) = -Cg_{\rho\nu}\eta^{\rho\mu}$$

Putting this back into (E.12) to get:

$$\Gamma^{\mu}{}_{\nu\sigma} = Cx^{\mu}g_{\nu\sigma} - 2Cx_{(\nu}g_{\sigma)\kappa}\eta^{\kappa\mu} = Cx^{\mu}g_{\nu\sigma} - 2Cx_{(\nu}\left(\delta^{\mu}_{\sigma)} - Cx_{\sigma)}x^{\mu}\right)$$

QED.

E.1.2 Proof of The Configuration Space Curvature in (6.10)

We want to show that the configuration space curvature in (6.10) (also (E.19)) is a solution:

$$R^{\mu}{}_{\nu\sigma\rho}dx^{\sigma}dx^{\rho} \otimes dx^{\nu} = \left(-\partial_{[\sigma}\Gamma^{\mu}{}_{\rho]\nu} + \Gamma^{\kappa}{}_{\nu[\sigma}\Gamma^{\mu}{}_{\rho]\kappa}\right)dx^{\sigma}dx^{\rho} \otimes dx^{\nu} \tag{E.18}$$

$$g_{\mu\nu} = \eta_{\mu\nu} - Cx_{\mu}x_{\nu}$$

$$\Gamma^{\mu}{}_{\nu\sigma} = Cx^{\mu}g_{\nu\sigma} - 2Cx_{(\nu}\left(\delta^{\mu}_{\sigma)} - Cx_{\sigma)}x^{\mu}\right) = Cx^{\mu}g_{\nu\sigma} - 2Cx_{(\nu}g_{\sigma)\kappa}\eta^{\kappa\mu}$$

$$R^{\mu}{}_{\nu\sigma\rho} = -C\left(\delta^{\mu}_{[\sigma} - Cx_{[\sigma}x^{\mu}\right)g_{\rho]\nu} \tag{E.19}$$

$$\nabla_{\sigma}dx^{\mu} = -\Gamma^{\mu}{}_{\nu\sigma}dx^{\nu} = -Cx^{\mu}dx_{\sigma}$$

Proof:

$$\begin{aligned} dx^{\sigma}dx^{\rho} \otimes \nabla_{[\sigma}\nabla_{\rho]}dx^{\mu} &= -dx^{\sigma}dx^{\rho} \otimes \nabla_{[\sigma}\left(\Gamma^{\mu}{}_{\rho]\nu}dx^{\nu}\right) \\ &= \left(-\partial_{[\sigma}\Gamma^{\mu}{}_{\rho]\nu} + \Gamma^{\kappa}{}_{\nu[\sigma}\Gamma^{\mu}{}_{\rho]\kappa}\right)dx^{\sigma}dx^{\rho} \otimes dx^{\nu} \\ &= R^{\mu}{}_{\nu\sigma\rho}dx^{\sigma}dx^{\rho} \otimes dx^{\nu} \end{aligned}$$

$$
\begin{aligned}
dx^\sigma dx^\rho \otimes \nabla_{[\sigma}\nabla_{\rho]} dx^\mu &= -dx^\sigma dx^\rho \otimes \nabla_{[\sigma}\left(Cx^\mu g_{\rho]\nu} dx^\nu\right) \\
&= -C\delta^\mu_{[\sigma} dx^\sigma dx^\rho \otimes dx_{\rho]} - Cx^\mu dx^\sigma dx^\rho \otimes \nabla_{[\sigma} dx_{\rho]} \\
&= -C\delta^\mu_{[\sigma} dx^\sigma dx^\rho \otimes dx_{\rho]} - Cx^\mu \eta_{\kappa[\rho} dx^\sigma dx^\rho \otimes \left(\nabla_{\sigma]} dx^\kappa\right) \\
&= -C\delta^\mu_{[\sigma} dx^\sigma dx^\rho \otimes dx_{\rho]} + C^2 x^\mu x^\kappa \eta_{\kappa[\rho} dx^\sigma dx^\rho \otimes dx_{\sigma]} \\
&= -C\delta^\mu_{[\sigma} dx^\sigma dx^\rho \otimes dx_{\rho]} + C^2 x^\mu x_{[\rho} dx^\sigma dx^\rho \otimes dx_{\sigma]} \\
&= -C\delta^\mu_{[\sigma} g_{\rho]\nu} dx^\sigma dx^\rho \otimes dx^\nu \\
&= -C\left(\delta^\mu_{[\sigma} - Cx_{[\sigma} x^\mu\right) g_{\rho]\nu} dx^\sigma dx^\rho \otimes dx^\nu
\end{aligned}
$$

Now the equation above for $R^\mu{}_{\nu\sigma\rho}$ is invariant under the interchanges:

$$
R^\mu{}_{\nu\sigma\rho} \to R^\mu{}_{\nu\sigma\rho} + x_\nu q^\mu{}_{\sigma\rho} + x_\sigma f^\mu{}_{\nu\rho} + x_\rho h^\mu{}_{\nu\sigma}
$$

where $q^\mu{}_{\sigma\rho}$, $f^\mu{}_{\nu\rho}$, and $h^\mu{}_{\nu\sigma}$ are all arbitrary. To fix the freedom we realize that R is a tensor therefore each index should be projected orthogonal to x:

$$
R^\mu{}_{\nu\sigma\rho} = -C\left(\delta^\mu_{[\sigma} - Cx_{[\sigma} x^\mu\right) g_{\rho]\nu}
$$

which is (E.19) and we can verify the usual symmetries of the Riemann tensor very easily. Also we verify that the curvature preserves the constraint $x_\mu dx^\mu = 0$:

$$
\begin{aligned}
dx^\sigma dx^\rho \otimes \nabla_{[\sigma}\nabla_{\rho]} x_\mu dx^\mu &= x_\mu dx^\sigma dx^\rho \otimes \nabla_{[\sigma}\nabla_{\rho]} dx^\mu \\
&= -Cx_\mu \left(\delta^\mu_{[\sigma} - Cx_{[\sigma} x^\mu\right) g_{\rho]\nu} = 0
\end{aligned}
$$

APPENDIX F

PROOF OF EQUATION (5.12)

We now show that the equation in (5.12) is true, i.e., $\left(D - \hat{D}\right)^2 \hat{y}^A = 0$ is equivalent to $\left[\Omega - Dr + \hat{d}r + r^2, \hat{y}^A\right] = 0$:

Proof:

$$\left(D - \hat{D}\right)^2 \hat{y}^A = \left(D^2 - D\hat{D} - \hat{D}D + \hat{D}^2\right)\hat{y}^A$$

$$\left(D\hat{D} + \hat{D}D\right)\hat{y}^A = \left[D\left(\omega_{AB}\hat{y}^A\Theta^B + r\right), \hat{y}^A\right] = \left[Dr, \hat{y}^A\right]$$

$$\begin{aligned}
\hat{D}^2\hat{y}^A &= \left[\hat{Q}, \left[\hat{Q}, \hat{y}^A\right]\right] = \hat{Q}\left(\hat{Q}\hat{y}^A - \hat{y}^A\hat{Q}\right) + \left(\hat{Q}\hat{y}^A - \hat{y}^A\hat{Q}\right)\hat{Q} \\
&= \left[\hat{Q}^2, \hat{y}^A\right]_- = \left[\left(\omega_{AB}\hat{y}^A\Theta^B + r\right)^2, \hat{y}^A\right]_- = \left[\left(\omega_{AB}\hat{y}^A\Theta^B\right)^2 + \left[\omega_{AB}\hat{y}^A\Theta^B, r\right] + r^2, \hat{y}^A\right]_-
\end{aligned}$$

$$2\left(\omega_{AB}\hat{y}^A\Theta^B\right)^2 = \left[\omega_{AB}\hat{y}^A\Theta^B, \omega_{CE}\hat{y}^C\Theta^E\right] = \left[\hat{y}^A, \hat{y}^C\right]\omega_{AB}\Theta^B\omega_{CE}\Theta^E = \omega_{AB}\Theta^A\Theta^B$$

$$\Longrightarrow \hat{D}^2\hat{y}^A = \left[\left[\omega_{AB}\hat{y}^A\Theta^B, r\right] + r^2, \hat{y}^A\right]_-$$

where $[A, B]_- = AB - BA$ for any A and B.

The curvature D^2 acting on Θ^A is:

$$D^2 \otimes \Theta^A = {R_B}^A \otimes \Theta^B$$

Thus the curvature D^2 acting on $\hat{y}^A$ is:

$$D^2\hat{y}^A = {R_B}^A\hat{y}^B$$

Knowing this we define Ω as the curvature D^2 acting on $\hat{y}^A$ as a commutator, namely:

$$\frac{1}{i\hbar}\left[\Omega,\hat{y}^A\right]=R_B{}^A\hat{y}^B$$

we can immediately write a solution for Ω knowing $\left[\hat{y}^A,\hat{y}^B\right]=i\hbar\omega^{AB}$, $\omega^{AB}\omega_{BC}=\delta^A_C$ and using the symmetries of the curvature tensor:

$$\Omega:=-\frac{1}{2}\omega_{AC}R_B{}^A\hat{y}^B\hat{y}^C$$

Thus we may rewrite the condition $\left(D-\hat{D}\right)^2\hat{y}^A=0$ as:

$$\left(D-\hat{D}\right)^2\hat{y}^A=\left[\Omega-Dr+\hat{d}r+r^2,\hat{y}^A\right]=0$$

APPENDIX G

A TECHNICAL NOTE ON THE FORM OF SOLUTIONS TO (5.11)

Technical Notes:

1. We can decompose $\hat{Q}$ into two parts, the first being the solution in the flat space case:

$$\hat{Q}_A \Theta^A = \omega_{AB} \hat{y}^A \Theta^B + r \tag{G.1}$$

   Equation (5.11) becomes:

$$\Omega - Dr + \hat{d}r + r^2 = 0 \tag{G.2}$$

$$\hat{d} := \left[\omega_{AB} \hat{y}^A \Theta^B, \cdot\right]$$

   The reason for this decomposition is that we have an argument why r should be more than quadratic in $\hat{y}$ (see below).

2. We ran the Fedosov algorithm a few times to help us see what form the ansatz should take. Also remember that when we require $\Omega - Dr + \hat{d}r + r^2 = 0$ modulo terms that commute with the $\hat{y}$'s.

<u>Argument of Why r Should Be At Least Cubic in $\hat{y}$</u>

Here we present an argument as to why r in G.3 only has terms that are cubic or higher powers in the $\hat{y}$'s.

Given:

$$\hat{D} = \left[\hat{Q}, \cdot\right] = \left[\hat{Q}_A \Theta^A, \cdot\right]$$

$$\hat{Q}_A = \sum_l Q_{j,l,AA_1 \cdots A_l} \hat{y}^{A_1} \cdots \hat{y}^{A_l}$$

we require:

$$\left(D-\hat{D}\right)^2\hat{y}^A=0$$

If we let:

$$\hat{Q}_A\Theta^A=\omega_{AB}\hat{y}^A\Theta^B+r \tag{G.3}$$

$$r=\sum_l r_{AA_1\cdots A_l}\Theta^A\hat{y}^{A_1}\cdots\hat{y}^{A_l}$$

If we want r to be globally defined for <u>all</u> manifolds we must define it out of non-degenerate tensors namely the metric, the symplectic form and the curvature. This is because Ω is degree 2 in the $\hat{y}$'s (i.e., $\Omega:=-\frac{1}{2}\omega_{AC}R_B{}^A\hat{y}^B\hat{y}^C$ has 2 $\hat{y}$'s). The degree is defined by:

$$\deg\left(a\right)=\left(\text{number of }\hat{y}\text{'s}\right)+2\left(\text{number of }\hbar\text{'s}\right)$$

A linear r would yield:

$$\underbrace{\Omega}_{2}-\underbrace{Dr}_{1}+\underbrace{\hat{d}r}_{0}+\underbrace{r^2}_{1}$$

and this cannot be zero for $\Omega\neq 0$. This means that r must have a quadratic term in it.

If r is quadratic ($r=\sum_{l=0}^2 r_{AA_1\cdots A_l}\Theta^A\hat{y}^{A_1}\cdots\hat{y}^{A_l}$), in general, there is no way to construct the degree 2 coefficient $r_{AA_1A_2}$ out of invariant tensors. Thus we require that r has terms that are cubic or higher powers in the $\hat{y}$'s. Fedosov mentions this fact also in Fedosov B. (1996).

For a specific manifold there might be an r that is quadratic. The argument above is meant for an r in a <u>general</u> construction for a <u>general</u> manifold and so we give a counterexample in the case when the manifold M is Euclidean space $(\mathbb{R}^n,\delta)$.

There is always the trivial solution to r:

$$r=-\frac{1}{2}\omega_{CB}\Gamma^C{}_A\hat{y}^A\hat{y}^B$$

where $\Gamma^C{}_A=\Gamma^C{}_{BA}\Theta^B$ are the Christoffel symbols associated to D. One can easily observe that this is a solution knowing $\left[\hat{y}^A,\hat{y}^B\right]=i\hbar\omega^{AB}$, $\omega^{AB}\omega_{BC}=\delta^A_C$ and using the symmetries of the Christoffel symbols. However, the Γ's are not necessarily globally defined and if we find an r in one coordinate patch on T^*M there is no guarantee that it will be well-defined in another. However, if $M=\mathbb{R}^n$ then this is a global r.

APPENDIX H

PROPERTIES OF THE GROENEWOLD-MOYAL STAR-PRODUCT

For all smooth phase-space functions f, g, h, and z, and any constants c_1, c_2, c_3, and c_4 the properties of the Groenewold-Moyal star-product are:

1. Linear:

$$(c_1 f + c_2 g) * (c_3 h + c_4 z) = c_1 c_3 (f * h) + c_1 c_4 (f * z) + c_2 c_3 (g * h) + c_2 c_4 (g * z)$$

2. Associative:

$$f * (g * h) = (f * g) * h$$

3. Hermitian:

$$\overline{f * g} = \bar{g} * \bar{f}$$

4. Identity:

$$f * 1 = f$$

5. The limit $\hbar \to 0$:

$$\lim_{\hbar \to 0} f * g = fg$$

$$\lim_{\hbar \to 0} [f, g]_* / (i\hbar) = [f, g]_P$$

6. To first order in $\hbar$ the commutator is the Poisson bracket:

$$[f, g]_* = i\hbar [f, g]_P + O\left(\hbar^2\right)$$

7. The isomorphism $\mathcal{W}$ between the star-algebra and the space of linear Hilbert space operators $\mathcal{Q}_{D,\hat{D}}$:

$$\mathcal{W}\left(\sum_l f_{A_1\cdots A_l} q^{A_1} * \cdots * q^{A_l}\right) = \sum_l f_{A_1\cdots A_l} \hat{q}^{A_1} \cdots \hat{q}^{A_l}$$

where $f_{A_1\cdots A_l}$ are constants.

8. Strongly Closed (Cyclic Trace):

$$Tr_* (f * g) = Tr_* (g * f)$$

where f and g are smooth phase-space functions which are only nonzero on regions of compact support.

APPENDIX I

THE DERIVATION OF THE PHASE-SPACE CONNECTION

In this appendix we construct a phase-space connection called the cotangent lift of a the Levi-Civita connection. This fixes a unique phase-space connection D.

Given the Levi-Civita connection ∇ on the configuration space M and subsequent curvature given the metric g on a general manifold M:

$$\nabla_\sigma f(x) = \frac{\partial f}{\partial x^\sigma} \tag{I.1}$$

$$\nabla_\sigma (dx^\mu) = -\Gamma^\mu{}_{\nu\sigma} dx^\nu$$

$$\nabla_\sigma \left(\frac{\partial}{\partial x^\mu}\right) = \Gamma^\nu{}_{\mu\sigma} \frac{\partial}{\partial x^\nu}$$

$$\nabla_{[\sigma}\nabla_{\rho]} (dx^\mu) = R^\mu{}_{\nu\sigma\rho} dx^\nu$$

where $R^\mu{}_{\nu\sigma\rho}$ is the Riemann tensor. Of course we have the conditions that ∇ preserves the metric g and is torsion-free:

$$\nabla_a g_{bc} = 0$$

$$\nabla_{[a}\nabla_{b]} f(x) = 0$$

for all functions $f(x)$. Together these uniquely fix ∇ and give the standard formula for the Christoffel symbols:

$$\Gamma^\rho{}_{\mu\nu} = -\frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{I.2}$$

where ∂_μ are the partial derivatives in some basis x^μ.

Define now a basis of covectors or forms $\Theta^B \in T^*T^*M$ of the cotangent bundle of the phase-space:

$$\Theta^B = (dx^\sigma, \alpha_\sigma)$$

where the dx's are the first n Θ's, the α's are the last n Θ's and they are defined to be:

$$\alpha_\mu := dp_\mu - \Gamma^\nu{}_{\mu\rho} dx^\rho p_\nu \tag{I.3}$$

To extend D to define $D \otimes \alpha_\mu$ we require that D preserves the symplectic form ω:

$$0 = D \otimes \omega = D \otimes (\alpha_\mu dx^\mu) \implies (D \otimes \alpha_\mu) dx^\mu = \left(\Gamma^\nu{}_{\mu\sigma} dx^\sigma \otimes \alpha_\nu\right) dx^\mu$$

where it can be shown that:

$$\omega = \alpha_\mu dx^\mu = dp_\mu dx^\mu \tag{I.4}$$

which can be proven by the torsion-free condition which tells us that $\Gamma^\nu{}_{[\mu\rho]} = 0$. Therefore we make the ansatz:

$$D \otimes \alpha_\mu := S_{\mu\rho\sigma} dx^\sigma \otimes dx^\rho + \Gamma^\nu{}_{\mu\sigma} dx^\sigma \otimes \alpha_\nu \tag{I.5}$$

where $S_{[\mu\rho]\sigma} = 0$.

We can fix $S_{\mu\rho\sigma}$ by requiring that the directional derivative $\mathcal{D}_v$ of a vector and covector in any direction $v^a(x)$ on the manifold is also a vector and covector respectively.

$$w_\mu(x) \text{ is a covector } \iff \mathcal{D}_v w_\mu := v^\rho \left(\partial_\rho w_\mu - \Gamma^\nu{}_{\mu\rho} w_\nu\right) \text{ is a covector}$$

$$w^\mu(x) \text{ is a vector } \iff \mathcal{D}_v w^\mu := v^\rho \left(\partial_\rho w^\mu + \Gamma^\mu{}_{\nu\rho} w^\nu\right) \text{ is a vector}$$

this means that for any $p_\mu = w_\mu(x)$ (i.e., any section in the cotangent bundle) the directional derivative of a covector is a covector. Then the following formula must hold:

$$\nabla_{[a}\nabla_{b]} w_c = R^d{}_{cab} w_d$$

for every w_μ by the definition of the Riemann tensor. This formula then fixes the skew part of equation (I.5) to be:

$$D\alpha_\mu := D \wedge \alpha_\mu = S_{\mu\rho\sigma} dx^\sigma dx^\rho + \Gamma^\nu{}_{\mu\sigma} dx^\sigma \alpha_\nu = -R^\nu{}_{\mu\rho\sigma} p_\nu dx^\sigma dx^\rho + \Gamma^\nu{}_{\mu\sigma} dx^\sigma \alpha_\nu$$

$$\implies S_{a[ce]} = -R^b{}_{ace} p_b$$

Therefore we need to solve for S_{ace} that satisfies the two conditions:

$$S_{[ac]e} = 0 \quad \& \quad S_{a[ce]} = -R^b{}_{ace} p_b$$

Let $S_{ace} := S^b{}_{ace} p_b$ and these conditions become:

$$S^b{}_{[ac]e} = 0 \quad \& \quad S^b{}_{a[ce]} = -R^b{}_{ace} \tag{I.6}$$

Using the first Bianchi identity, the solution to this equation is:

$$S^b{}_{ace} = -\frac{4}{3} R^b{}_{(ac)e} \tag{I.7}$$

Therefore:

$$D \otimes \alpha_\mu := -\frac{4}{3} R^\psi{}_{(\mu\sigma)\beta} p_\psi dx^\beta \otimes dx^\sigma + \Gamma^\nu{}_{\mu\sigma} dx^\sigma \otimes \alpha_\nu$$

The phase-space connection is:

$$Dx^\mu := dx^\mu \tag{I.8}$$

$$Dp_\mu := dp_\mu$$

$$D \otimes dx^\mu = -\Gamma^\mu{}_{\sigma\nu} dx^\nu \otimes dx^\sigma$$

$$D \otimes \alpha_\mu = \Theta^B \otimes D_B \alpha_\mu := -\frac{4}{3} R^\psi{}_{(\mu\sigma)\beta} p_\psi dx^\beta \otimes dx^\sigma + \Gamma^\nu{}_{\mu\sigma} dx^\sigma \otimes \alpha_\nu$$

$$\alpha_\mu := dp_\mu - \Gamma^\nu{}_{\mu\rho} dx^\rho p_\nu$$

which is the connection in (5.4) and the corresponding curvature:

$$D^2 x^\mu = 0 \tag{I.9}$$

$$D^2 p_\mu = 0$$

$$D^2 \otimes dx^\mu = dx^\sigma dx^\rho \otimes R^\mu{}_{\nu\sigma\rho} dx^\nu$$

$$D^2 \otimes \alpha_\mu = \frac{4}{3} dx^\sigma \left(C^\psi{}_{\mu\beta\nu\sigma} p_\psi dx^\nu + R^\nu{}_{(\mu\beta)\sigma} \alpha_\nu \right) \otimes dx^\beta - R^\nu{}_{\mu\sigma\beta} dx^\sigma dx^\beta \otimes \alpha_\nu$$

which is the curvature in (5.6) where $C^c{}_{abes} := \nabla_s R^c{}_{(ab)e}$ and according to (5.1) the formula for the curvature is:

$$R^\mu{}_{\nu\sigma\rho} = -\partial_{[\sigma} \Gamma^\mu{}_{\rho]\nu} + \Gamma^\kappa{}_{\nu[\sigma} \Gamma^\mu{}_{\rho]\kappa} \tag{I.10}$$

We can extend to higher order tensors by using the Leibnitz rule and the fact that D and ∇ commute with contractions.

APPENDIX J

THE WEYL TRANSFORM

In this appendix we review the definition and properties of the Weyl quantization map $\mathcal{W}$ and its inverse $\mathcal{W}^{-1}$. The generalization to the n-dimensional case is straightforward once you know how the 1-dimensional case works so we consider only the 1-dimensional case. Let $\hat{x}$ and $\hat{p}$ be Hilbert space operators that satisfy the commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar \quad , \quad [\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$$

The definitions of $\mathcal{W}$ and $\mathcal{W}^{-1}$ are:

$$\mathcal{W}\left(f\left(x, p\right)\right) := \int d\xi d\eta \; e^{-i(\xi\hat{x}+\eta\hat{p})} \tilde{f}\left(\xi, \eta\right) \tag{J.1}$$

$$\mathcal{W}^{-1}\left(\hat{f}\left(\hat{x}, \hat{p}\right)\right) := \hbar \int dy \left\langle x + \hbar y/2 | \hat{f} | x - \hbar y/2 \right\rangle e^{-ipy} \tag{J.2}$$

where $f\left(x, p\right)$ is a function on the phase-space ($C^{\infty}\left(T^{*}M\right)$) and $\tilde{f}\left(\xi, \eta\right)$ is its Fourier transform (see Hancock J. *et al* 2004 and Hirshfeld A. and Henselder P. 2002a).

Important identities are for all $f \in C^{\infty}\left(T^{*}M\right)$ and $\hat{f} \in \mathcal{Q}_{D,\hat{D}}$:

$$\mathcal{W}^{-1}(\mathcal{W}\left(f\right)) = f \quad , \quad \mathcal{W}(\mathcal{W}^{-1}(\hat{f})) = \hat{f}$$

and for 1-dimensions:

$$\mathcal{W}\left(ax + bp\right)^{n} = \left(a\hat{x} + b\hat{p}\right)^{n}$$

What $\mathcal{W}$ gives is the average of all orderings for each monomial e.g. $\mathcal{W}(x^2p) = \frac{1}{3}(\hat{x}^2\hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2)$. It achieves this in the following way. First note that $\mathcal{W}\left(e^{ax+bp}\right) = e^{a\hat{x}+b\hat{p}}$. Since most functions can be defined uniquely by sums of terms like these with constant coefficients (i.e., Fourier components) it gives this unique ordering to all functions known as (symmetric) Weyl ordering.

A key fact that one should know of this transform is that to each operator $\hat{f}$ there exists a unique phase-space function $f(q)$ where $q = (x, p)$ and with the property that:

$$\hat{f} := \mathcal{W}(f) = \sum_l f_{A_1\cdots A_l}\hat{q}^{A_1}\cdots\hat{q}^{A_l}$$

becomes:

$$f = \mathcal{W}^{-1}(\hat{f}) = \sum_l f_{A_1\cdots A_l} q^{A_1} * \cdots * q^{A_l}$$

in a mechanical way by simply replacing each $\hat{q}$ with q and placing Groenewold-Moyal stars between each of them as is done above. The coefficients $f_{A_1\cdots A_l}$ are required to be symmetric in $(A_1 \cdots A_l)$.

Other properties of the Weyl transform are:

1. The trace transforms to an integral:

$$Tr_{tr}(\hat{f}) = Tr_{tr*}(f) := \frac{1}{(2\pi\hbar)^n}\int d^npd^nx\ f = \frac{1}{(2\pi\hbar)^n}\int d^{2n}q\ f \tag{J.3}$$

   where the trace of $\hat{f}$ over the translational degrees of freedom on a Hilbert space operator is defined as:

$$Tr_{tr}(\hat{f}) := \int d^nx\ \langle x|\hat{f}|x\rangle = \int d^np\ \langle p|\hat{f}|p\rangle$$

2. Hermitian conjugation becomes complex conjugation denoted by the bar:

$$\mathcal{W}^{-1}(\mathcal{W}(f)^{\dagger}) = \bar{f}\ .$$

The importance of this transform cannot be understated. It relates everything one does in a operator formalism of quantum mechanics to a phase-space formalism, vis a vis this algebra-isomorphism.

BIBLIOGRAPHY

Barton G. 1963 *Introduction to Advanced Field Theory* (New York: Interscience).

Bayen F. *et al* 1978 *Ann. Physics* **111**, 61.

Basart H. *et al* 1984 *Lett. Math. Phys.* **8**, 483.

Bertelson M. *et al* 1997 *Class. Quantum Grav.* **14**, A93.

Birrell N. and Davies P. 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press).

Bordemann M. *et al* 2003 *J. Funct. Anal.* **199** (1), 1 *Preprint* math.QA/9811055.

Bordemann M. *et al* 1998 *Lett. Math. Phys.* **45**, 49.

Bordemann M. and Waldmann S. 1998 *Commun. Math. Phys.* **195**, 549.

Buchholz, D. 2000 *Proc. XIIIth Int. Congress on Mathematical Physics (London)* Preprint math-ph/0011044.

Brunetti R. and Fredenhagen K. 2004 *Algebraic approach to Quantum Field Theory*, to appear on Elsevier Encyclopedia of Mathematical Physics, math-ph/0411072.

Brattelli O. and Robinson D. 1979 *Operator Algebras and Quantum Statistical Mechanics* (New York: Springer-Verlang).

Connes A. 1992a *Noncommutative Geometry* (San Diego: Academic Press).

Connes A. *et al* 1992b *Lett. Math. Phys.* **24**, 1.

Dito G. 2002 *Proc. Int. Conf. of 68ème Rencontre entre Physiciens Théoriciens et Mathématiciens on Deformation Quantization I (Strasbourg)*, (Berlin: de Gruyter) p 55 *Preprint* math.QA/0202271.

Dito G. and Sternheimer D. 2002 *Proc. Int. Conf. of 68ème Rencontre entre Physiciens Théoriciens et Mathématiciens on Deformation Quantization I (Strasbourg)*, (Berlin: de Gruyter) p 9 *Preprint* math.QA/0201168.

Dütsch M. and Fredenhagen K. 2000 *Proc. Conf. Math. Physics in Mathematics and Physics (Siena)*, Preprint hep-th/0101079.

Fedosov B. 1996 *Deformation Quantization and Index Theory* (Berlin: Akademie).

Frønsdal C. 1965 *Rev. Mod. Phys.* **37**, 221.

Frønsdal C. 1973 *Phys. Rev.* D **10**, 2, 589.

Frønsdal C. 1975a *Phys. Rev.* D **12**, 12, 3810.

Frønsdal C. 1975b *Phys. Rev.* D **12**, 12, 3819.

Gerstenhaber M. 1964 *Ann. Math.* **79**, 59.

Giulini D. 2003 *Quantum Gravity: From Theory to Experimental Search*, (Bad Honnef, Germany: Springer) p 17.

Gozzi E. and Reuter M. 1994 *Int. J.Mod.Phys.* **A9**, 32, 5801-5820 *Preprint* hep-th/0306221.

Groenewold H. 1946 *Physica* **12**, 405.

Gadella M. *et al* 2005 *J. Geom. Phys.* **55**, 316.
Haag R. 1992 *Local Quantum Physics* (New York: Springer-Verlang).
Hakioğlu T. and Dragt A. 2001 *J. Phys. A: Math. Gen.* **34** 6603.
Hancock J. *et al* 2004 *Eur. J. Phys.* **25,** 525.
Hawking S. and Ellis G. 1973 *Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press).
Hirshfeld A. 2003 *Proc. IV Int. Conf. on Geometry, Integrability and Quantization (Varna, Bulgaria)* (Coral Press, Sofia), p 11.
Hirshfeld A. and Henselder P. 2002a *Am. J. Phys.* **70** (5), 537.
Hirshfeld A. and Henselder P. 2002b *Annals Phys.* **298**, 382.
Hirshfeld A. and Henselder P. 2002c *Annals Phys.* **302**, 59.
Hirshfeld A. and Henselder P. 2003 *Annals Phys.* **308**, 311.
Hirshfeld A. *et al* 2005 *Annals Phys.* **317**, 107.
Hirshfeld A. *et al* 2004 *Annals Phys.* **314**, 75.
Jancel R. 1963 *Foundations of Classical & Quantum Statistical Mechanics* (Pergamon Press Oxford).
Kontsevich M. 2003 *Lett. Math. Phys.* **66**, 157.
Landau L. 1938 *Statistical Physics* (London: Oxford Press).
Moyal J. 1949 *Proc. Cambridge Phil. Soc.* **45**, 99.
Norton J. 1993 *Rep. Prog. Phys.* **56**, 791.
Pauli W. 1940 *Phys. Rev.* **58**, 716.
Rovelli C. 2000 *Lect.Notes Phys.* **541**, 277.
Tillman P. and Sparling G. 2006a Fedosov Observables on Constant Curvature Manifolds and the Klein-Gordon Equation, *Preprint* gr-qc/0603017.
Tillman P. and Sparling G. 2006b *J. Math. Phys.* **47**, 052102.
Tillman P. 2006a to appear in *Proc. of II Int. Conf. on Quantum Theories and Renormalization Group in Gravity and Cosmology*, Preprint gr-qc/0610141.
Tillman P. 2006b *Deformation Quantization: From Quantum Mechanics to Quantum Field Theory*, Preprint gr-qc/0610159.
Waldmann S. 2001 *On the Representation Theory of Deformation Quantization*, Preprint math.QA/0107112.
Waldmann S. 2004 *States and Representations in Deformation Quantization*, Preprint math.QA/0408217.
Wald R. 1984 *General Relativity*, (Chicago: University of Chicago Press).
Wald R. 1994 *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Chicago: University of Chicago Press).
Weinstein A. 1995 Séminaire Bourbaki, Vol. 1993/94, Astérisque No. 227, Exp. No. 789, 5, 389-409.
Weinberg S. 1995 *The Quantum Theory of Fields* (New York: Cambridge University Press).
Weyl H. 1931 *The Theory of Groups and Quantum Mechanics*, (Dover: New York).
Wigner E. 1932 *Phys. Rev.* **40**, 749.
Witten E. 1988 *Commun. Math. Phys.* **117**, 353.
Woodhouse N. 1980 *Geometric Quantization* (New York: Oxford University Press).
Zachos C. and Curtright T. 1999 *Prog. Theor. Phys. Suppl.* **135**, 244.
Zachos C. 2002 *Int. J. Mod. Phys.* **A17**, 297.